Binary Random Process: Let

$$x(t) = \sum_{i=-\infty}^{\infty} a_i g(t - iT + \tau)$$

where the a_i's and τ are statistically independent random variables such that

$$\begin{aligned} p_{a_i}(\alpha) &= 1/2\delta(\alpha - 1) + 1/2\delta(\alpha + 1), \ \forall i \in \mathbb{Z} \\ p_\tau(\alpha) &= \begin{cases} 1/T & ; \ 0 \le \alpha < T \\ \\ 0 & ; \ \text{elsewhere} \end{cases} \end{aligned}$$

and $g(t)$ is a deterministic "pulse" which is 0 outside of the interval $[0, T]$. Typical sample functions are sketched on figures 1 and 2.

1. Mean function:

$$\begin{aligned} m_x(t) &= E[x(t)] \\ &= E[\sum_{i=-\infty}^{\infty} a_i g(t - iT + \tau)] \\ &= \sum_{i=-\infty}^{\infty} E[a_i g(t - iT + \tau)] \\ &= \sum_{i=-\infty}^{\infty} \underbrace{E[a_i]}_{0} E[g(t - iT + \tau)] \\ &= 0 \end{aligned}$$

   independent of time.

2. Autocorrelation function:

$$\begin{aligned}
\mathscr{R}_x(t,s) &= E[x(t)x(s)] \\
&= E\Big[\Big(\sum_{i=-\infty}^{\infty} a_i g(t-iT+\tau)\Big)\Big(\sum_{i=-\infty}^{\infty} a_i g(s-iT+\tau)\Big)\Big] \\
&= E\Big[\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} a_i a_j g(t-iT+\tau)g(s-jT+\tau)\Big] \\
&= \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} E[a_i a_j]E[g(t-iT+\tau)g(s-jT+\tau)] \\
&= \sum_{i=-\infty}^{\infty} \underbrace{E[a_i^2]}_{1} E[g(t-iT+\tau)g(s-iT+\tau)] \\
&= \sum_{i=-\infty}^{\infty} E[g(t-iT+\tau)g(s-iT+\tau)]
\end{aligned}$$

The expected value inside the above sum is expanded as follows:

$$\begin{aligned}
&E[g(t-iT+\tau)g(s-iT+\tau)] \\
&\qquad = \int_0^T g(t-iT+\alpha)g(s-iT+\alpha)p_\tau(\alpha)\mathrm{d}\alpha \\
&\qquad = \frac{1}{T}\int_{t-iT}^{t-(i-1)T} g(u)g(s-t+u)\mathrm{d}u
\end{aligned}$$

where, in the last line, we use the change of variables

$$\begin{aligned}
u = t-iT+\alpha \Leftrightarrow\ & \alpha = u-t+iT \\
& \mathrm{d}\alpha = \mathrm{d}u \\
& \alpha = 0 \Rightarrow u = t-iT \\
& \alpha = T \Rightarrow u = t-(i-1)T
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathscr{R}_x(t,s) &= \sum_{i=-\infty}^{\infty}\frac{1}{T}\int_{t-iT}^{t-(i-1)T} g(u)g(s-t+u)\mathrm{d}u \\
&= \frac{1}{T}\int_{-\infty}^{\infty} g(u)g(s-t+u)\mathrm{d}u \\
&= \frac{1}{T}R_g(s-t)
\end{aligned}$$

where we define:

$$
\begin{aligned}
R_g(t) &= \int_{-\infty}^{\infty} g(u)g(u-t)\mathrm{d}u \\
&= g(t) * g(-t)
\end{aligned}
$$

Hence $\mathscr{R}_x(t,s)$ is independent of the time origin.

Remark. The function $R_g(t)$ is called *(deterministic) autocorrelation function of the energy signal* $g(t)$. We have then showed that the autocorrelation function $\mathscr{R}_x(t,s)$ of the binary random process $x(t)$ is proportional to the deterministic autocorrelation function $R_g(t)$ at $s-t$. Even though they bear the same name, $\mathscr{R}_x(t,s)$ and $R_g(t)$ are not to be confused; $\mathscr{R}_x(t,s)$ is used with random processes and $R_g(t)$ is used with deterministic (energy) signals.

Case 1 (Square pulse). The pulse is shown in figure 1(a). Typical sample functions are sketched on figures 1(b) - 1(e), and the corresponding autocorrelation function is plotted on figure 1(f).

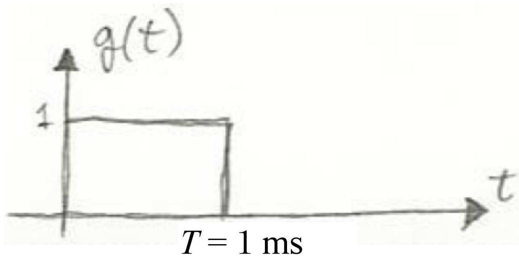
Case 2 (Raised-sine pulse). The pulse is shown in figure 2(a). Typical sample functions are sketched on figures 2(b) - 2(e), and the corresponding autocorrelation function is plotted on figure 2(f). Notice that the maximum value 0.375 in the autocorrelation function corresponds to:

$$
\begin{aligned}
\frac{1}{T}\int_0^T g^2(t)\mathrm{d}t &= \frac{1}{1\text{ ms}}\int_0^{1\text{ ms}} \left(\frac{1-\cos(2\pi(1\text{ kHz})t)}{2}\right)^2 \mathrm{d}t \\
&= 0.375
\end{aligned}
$$

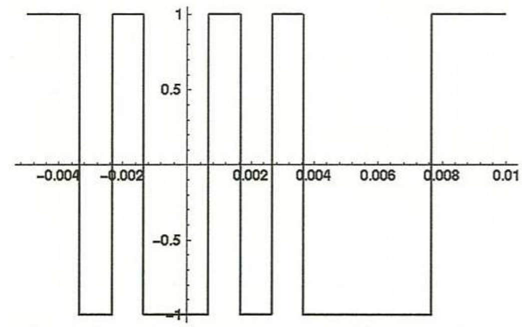
We take the Fourier transform of the autocorrelation functions of the binary random processes sketched in figures 1 and 2 to estimate the bandwidth requirements of an *average* sample function.

From above we have:

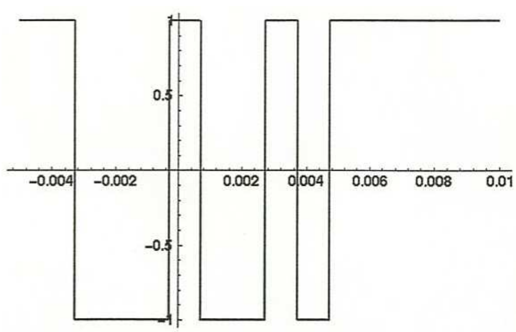
$$
\begin{aligned}
\mathscr{R}_x(\tau) &= \frac{1}{T}R_g(\tau) \\
&= \frac{1}{T}g(\tau) * g(-\tau)
\end{aligned}
$$



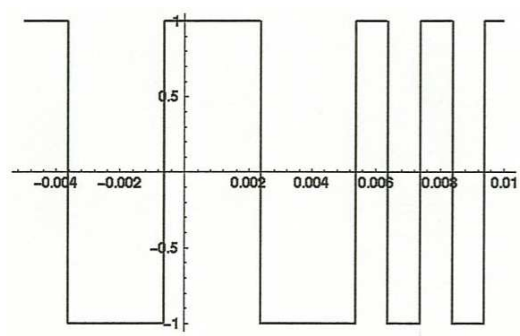
(a) Square pulse in binary random process



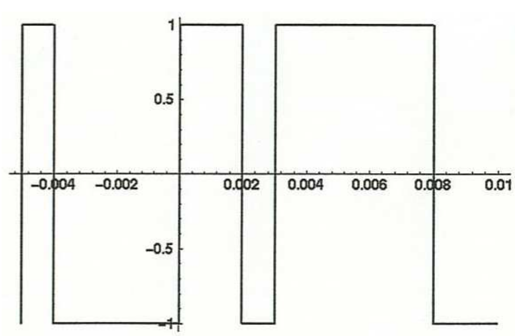
(b) typical sample function



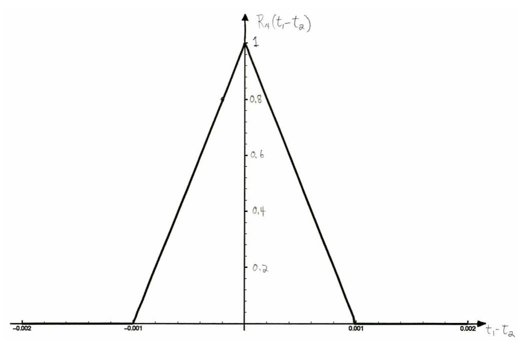
(c) other typical sample function



(d) yet another sample function

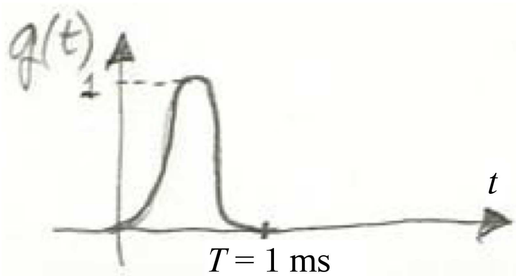


(e) yet another one more sample function

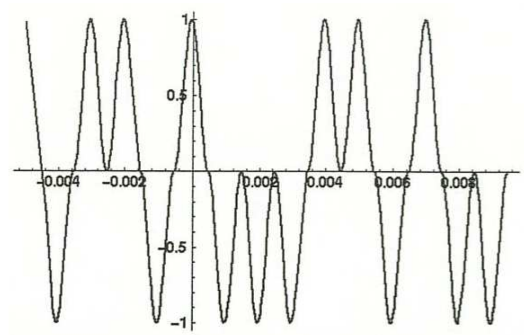


(f) autocorrelation function

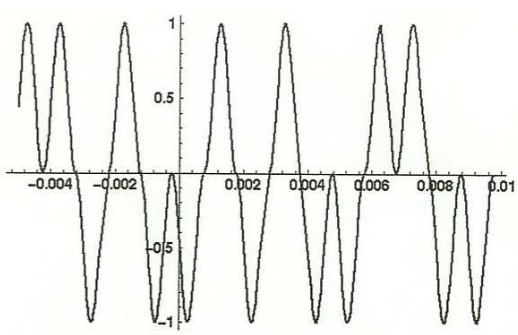
Figure 1: Binary random process with square pulses, $T = 1$ ms.



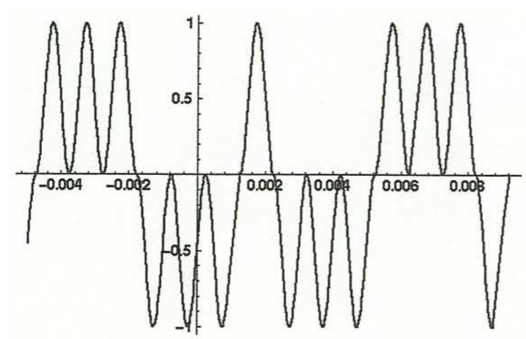
(a) Raised-sine pulse in binary random process



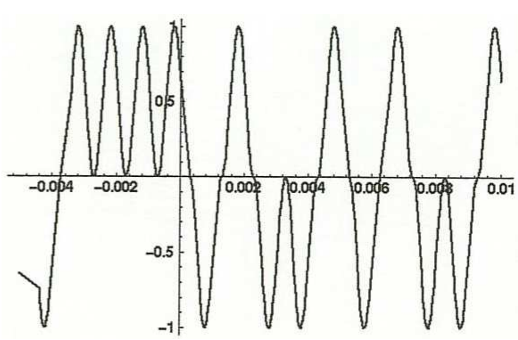
(b) typical sample function



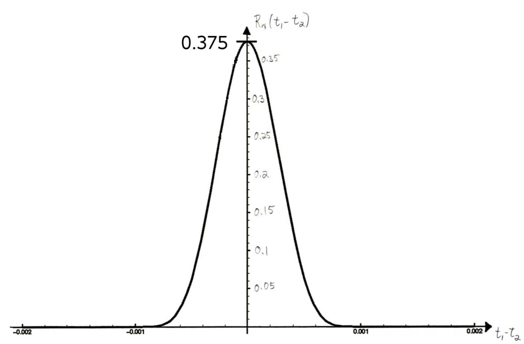
(c) other typical sample function



(d) yet another sample function



(e) yet another one more sample function



(f) autocorrelation function

Figure 2: Binary random process with raised sine pulses, $T = 1$ ms.

By definition and the properties of the Fourier transform we have:

$$\begin{aligned} S_x(f) &= \mathscr{F}(\mathscr{R}_x(\tau)) \\ &= \frac{1}{T}|G(f)|^2 \end{aligned}$$

where $G(f) = \mathscr{F}(g(t))$ denotes the Fourier transform of the pulse $g(t)$. In the following we use:

$$\begin{aligned} \text{sinc}(x) &\triangleq \frac{\sin(\pi x)}{\pi x} \\ \text{rect}(x) &\triangleq \begin{cases} 1 & ; \quad -1/2 < x < 1/2 \\ 0 & ; \quad \text{elsewhere} \end{cases} \\ \Delta(x) &\triangleq \begin{cases} 1 - 2|x| & ; \quad -1/2 < x < 1/2 \\ 0 & ; \quad \text{elsewhere} \end{cases} \end{aligned}$$

- For the rectangular pulse we use the Fourier transform pair:

  $$\text{rect}(t/(1\text{ ms})) \longleftrightarrow (1\text{ ms})\,\text{sinc}((1\text{ ms})\ f) = (1\text{ ms})\,\text{sinc}(f/1000)$$

  It follows that

  $$|G(f)| = (1\text{ ms})\,|\text{sinc}(f/1000)|$$

  for the rectangular pulse.[1]

- For the triangular pulse we use the Fourier transform pair

  $$\Delta(t/(1\text{ ms})) \longleftrightarrow (0.5\text{ ms})\,\text{sinc}^2((0.5\text{ ms})\ f)$$

  This is $G(f)$ for the triangular pulse.

- For the raised-cosine pulse we notice that

  $$\begin{aligned} g(t + 0.5\text{ ms}) &= \text{rect}(t/(1\text{ ms}))\left(\frac{1 + \cos(2\pi t/(1\text{ ms}))}{2}\right) \\ &= \frac{1}{2}\text{rect}(t/(1\text{ ms})) + \frac{1}{2}\text{rect}(t/(1\text{ ms}))\ \cos(2\pi t/(1\text{ ms})) \end{aligned}$$

[1] recall that the pulses go from 0 to $T = 1$ ms, whereas the above goes from -0.5 ms to +0.5 ms. The time delay affects the phase of the Fourier transform only; it has no effect on the magnitude, which is all that is required here.

Using the property of modulation on the Fourier transform pair

$$\frac{1}{2}\text{rect}(t/(1\text{ ms})) \longleftrightarrow (0.5\text{ ms})\ \text{sinc}((1\text{ ms})\ f) = (0.5\text{ ms})\ \text{sinc}(f/1000)$$

we obtain

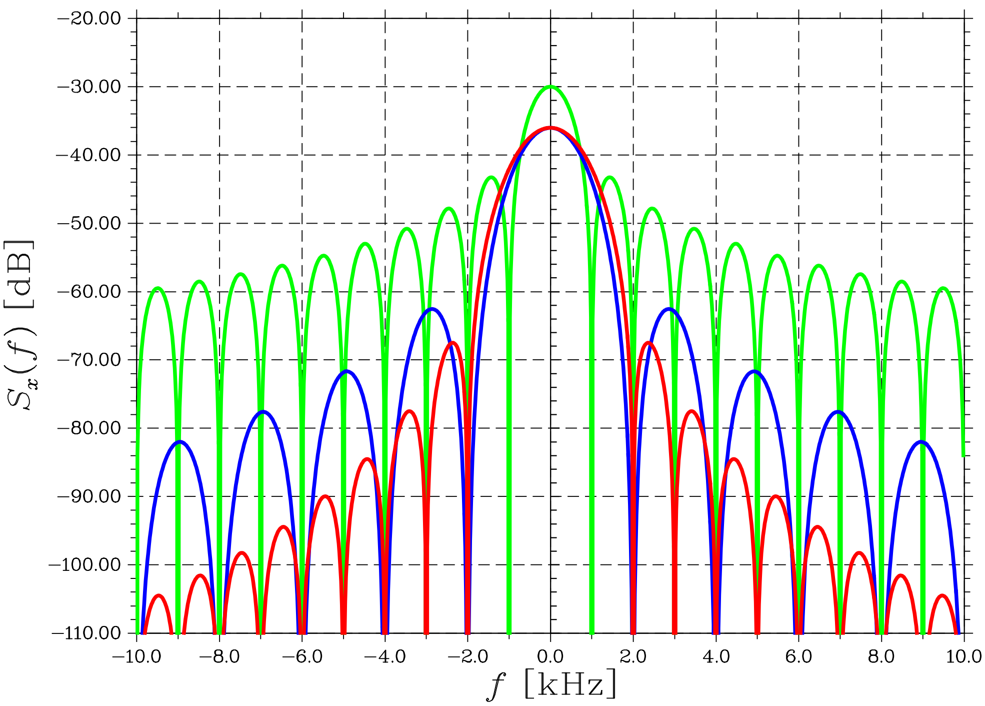
$$\frac{1}{2}\text{rect}(t/(1\text{ ms}))\ \cos(2\pi t/(1\text{ ms})) \longleftrightarrow (0.25\text{ ms})\text{sinc}((1.0\text{ ms})(f-(1\text{ kHz})))+ (0.25\text{ ms})\text{sinc}((1.0\text{ ms})(f+(1\text{ kHz}))) = (0.25\text{ ms})\text{sinc}((f-1000)/1000)+ (0.25\text{ ms})\text{sinc}((f+1000)/1000)$$

It follows that

$$|G(f)| = \Big|(0.5\text{ ms})\text{sinc}(f/1000)+ (0.25\text{ ms})\text{sinc}((f-1000)/1000)+ (0.25\text{ ms})\text{sinc}((f+1000)/1000)\Big|$$

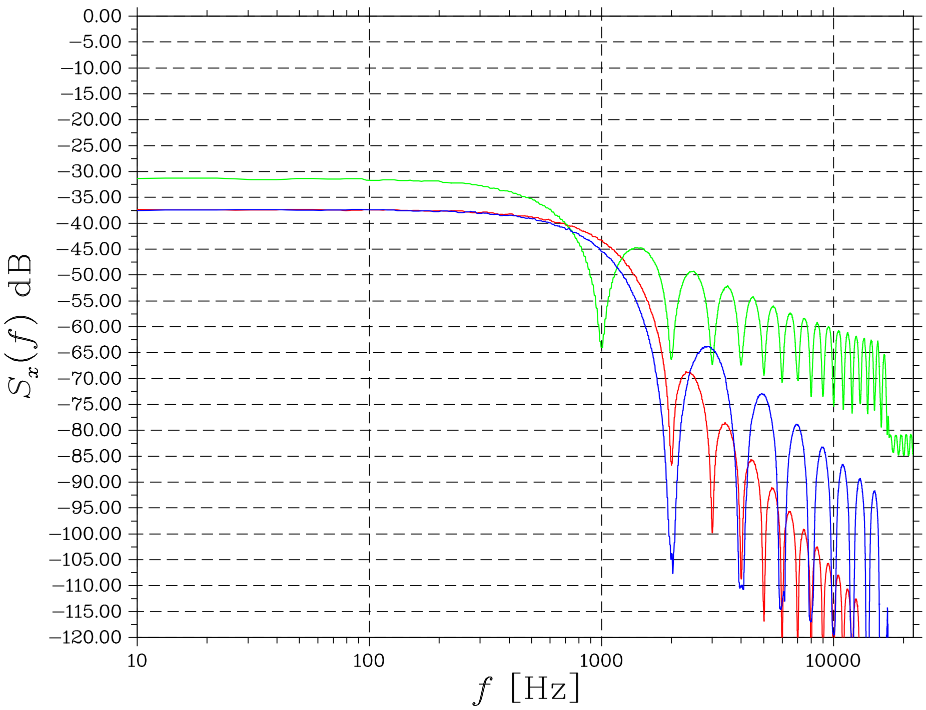
for the raised cosine pulse.

Figure 3 shows sketches of the Power spectral densities of the binary random processes for the cases of rectangular pulse (green curve), triangular pulse (blue curve) and raised cosine pulse (red curve). Experimental measurements were taken which confirm the above results; refer to figure 4.



−20.00
−30.00
−40.00
−50.00
−60.00
−70.00
−80.00
−90.00
−100.00
−110.00
$S_x(f)$ [dB]
−10.0
−8.0
−6.0
−4.0
−2.0
0.0
2.0
4.0
6.0
8.0
10.0
f [kHz]

Figure 3:


0.00
−5.00
−10.00
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−25.00
−30.00
−35.00
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−45.00
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−55.00
−60.00
−65.00
−70.00
−75.00
−80.00
−85.00
−90.00
−95.00
−100.00
−105.00
−110.00
−115.00
−120.00
10
100
1000
10000
$S_x(f)$ dB
f [Hz]

Figure 4: