Solution #1: Refer to the solution of problem 3.1.

Solution **#2**

(a) From the formulas

$$\underline{m}_y = \underline{m}_x \times A^T + \underline{b}$$

$$\Lambda_y = A \times \Lambda_x \times A^T$$

where A and $\underline{b}$ are identified in

$$\underline{y} = \underline{x} \times A^T + \underline{b}$$

$$= \underline{x} \times \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} + (0, 1)$$

we immediately obtain

$$\underline{m}_y = (1, 0) \times \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} + (0, 1)$$

$$= (1, 2)$$

$$\Lambda_y = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} 9 & -4 \\ -4 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 21 & -11 \\ -11 & 21 \end{pmatrix}$$

(b) The hatched region corresponds to

$$0 < \underbrace{2y_1 + 3y_2}_{z} < 24$$

where $z \triangleq 2y_1 + 3y_2 = (y_1, y_2) \times (2, 3)^T$ is Gaussian since $\underline{y}$ is Gaussian. The mean and variance of z are given by the formulas:

$$\bar{z} = \underline{m_y} \times (2, 3)^T$$

$$Var(z) = (2, 3) \times \Lambda_y \times (2, 3)^T$$

and we immediately obtain

$$\bar{z} = (1, 2) \times (2, 3)^T = 8$$

$$Var(z) = (2, 3) \times \begin{pmatrix} 21 & -11 \\ -11 & 21 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$= 141$$

Finally we obtain

$$P(0 \leq z < 24) = Q\left(\frac{0-8}{\sqrt{141}}\right) - Q\left(\frac{24-8}{\sqrt{141}}\right)$$

$$= 0.660837\ldots$$

$$\boxed{\begin{aligned} &\approx 1 - 0.242 - 0.0888 \\ &= 0.6692 \end{aligned}}$$ with tables

With $\underline{m_y} = (2, 1)$ and $\Lambda_y = \begin{pmatrix} 25 & -10 \\ -10 & 25 \end{pmatrix}$ we would obtain $\bar{z} = 7$, $\sigma_z^2 = 115$. Finally

$$P(0 \leq z < 24) = Q\left(\frac{-7}{\sqrt{115}}\right) - Q\left(\frac{24-7}{\sqrt{115}}\right)$$

$$= 0.6865885\ldots$$

Solution #3

$$R_{x}(t_1,t_2) = E\Big[\big(A\sin(2\pi f t_1+\theta)+y(t_1)\big)\big(A\sin(2\pi f t_2+\theta)+y(t_2)\big)\Big]$$

$$= E\Big[A^2\sin(2\pi f t_1+\theta)\sin(2\pi f t_2+\theta) + Ay(t_1)\sin(2\pi f t_2+\theta) + Ay(t_2)\sin(2\pi f t_1+\theta) + y(t_1)y(t_2)\Big]$$

$$= E\left[\frac{A^2}{2}\Big(\cos(2\pi f(t_1-t_2)) - \cos(2\pi f(t_1+t_2)+2\theta)\Big)\right] + E[A]\,E[y(t_1)]\,E[\sin(2\pi f t_2+\theta)] + E[A]\,E[y(t_2)]\,E[\sin(2\pi f t_2+\theta)] + E[y(t_1)\,y(t_2)]$$

(i) $E[\sin(2\pi f t_2+\theta)] = \int_0^{2\pi} \sin(2\pi f t_2+\alpha)\,d\alpha$
$= 0 = E[\sin(2\pi f t_1+\theta)]$

(ii) $E[y(t_1)\,y(t_2)] = R_y(t_2-t_1) = R_y(t_1-t_2)$

(iii) $E\left[\frac{A^2}{2}\cos(2\pi f(t_1-t_2))\right] = \frac{\cos(2\pi f(t_1-t_2))}{2}E[A^2]$
$= \frac{b}{2}\cos(2\pi f(t_1-t_2))$

(iv) $E\left[\frac{A^2}{2}\cos(2\pi f(t_1+t_2)+2\theta)\right]$
$= \frac{1}{2}E[A^2]\,E[\cos(2\pi f(t_1+t_2)+2\theta)]$
$= \frac{b}{2}\int_0^{2\pi}\frac{\cos(2\pi f(t_1+t_2)+2\alpha)}{2\pi}\,d\alpha = 0$

It follows that

$$R_Y(t_1, t_2) = \frac{b}{2} \cos(2\pi f(t_1 - t_2)) + R_y(t_1 - t_2)$$