Solution #1

The mean and variance of x_i are obtained as follows:

$$\bar{x} = \bar{x}_i = (-1)(1/2) + (0)(1/4) + (1)(1/4)$$
$$= -1/4$$

$$\overline{x^2} = \overline{x_i^2} = (-1)^2(1/2) + (0)^2(1/4) + (1)^2(1/4)$$
$$= 3/4$$

$$\sigma_x^2 = Var(x_i) = 3/4 - (-1/4)^2 = 11/16 = 0.6875$$

(a) From the WLLN we have

$$P(|m - \bar{x}| \geq \varepsilon) \leq \frac{\sigma_x^2}{N\varepsilon^2}$$

$$\Rightarrow \quad P(|m - \underset{-1/4}{\bar{x}}| < \varepsilon) = 1 - P(|m - \bar{x}| \geq \varepsilon)$$
$$\geq \underbrace{1 - \sigma_x^2/N\varepsilon^2}_{95\%} \Rightarrow \varepsilon = \sqrt{\frac{\sigma_x^2}{N(1-0.95)}}$$
$$= 0.11726$$

The range is then given by

$$|m - (-1/4)| < 0.11726$$

Or equivalently

$$\underbrace{-0.11726 - 1/4}_{-0.36726} < m < \underbrace{0.11726 - 1/4}_{-0.13274}$$

We are therefore 95% certain that m lies in the interval $[-0.36726, -0.13274]$.

(b) From the CLT we have

$$P(|m - \underset{-1/4}{\bar{Y}}| < \varepsilon) \approx \underbrace{1 - 2Q\left(\frac{\varepsilon\sqrt{N}}{\sigma_Y}\right)}_{0.95}$$

$$\Rightarrow Q\left(\varepsilon\sqrt{N/\sigma_Y^2}\right) \approx 0.025$$

$$\Rightarrow \varepsilon\sqrt{N/\sigma_Y^2} \approx 1.959963$$

$$\Rightarrow \varepsilon \approx \frac{1.959963}{\sqrt{1000/0.6875}} = 0.05139$$

The range is then given by

$$|m - (-1/4)| < 0.05139$$

or equivalently

$$\underbrace{-0.25 - 0.05139}_{-0.30139} < m < \underbrace{0.05139 - 0.25}_{-0.19861}$$

We are therefore 95% certain that m lies in the interval $[-0.30139, -0.19861]$

Solution #2

(a) $m_X(t) = E[X(t)]$

$$= E[10 \sin(2\pi f t)]$$

$$= \int_0^{100} \frac{10}{100} \sin(2\pi \alpha t)\, d\alpha$$

$$= \frac{1}{20\pi t} \Big[\cos(2\pi \alpha t)\Big]_{100}^{0}$$

$$= \frac{1 - \cos(200\pi t)}{20\pi t}$$

$t \neq 0$, but the formula gives the right answer if $t = 0$.

$$R_X(t_1, t_2) = E[X(t_1) X(t_2)]$$

$$= E[100 \sin(2\pi f t_1) \sin(2\pi f t_2)]$$

$$= 50\, E[\cos(2\pi f(t_1 - t_2)) - \cos(2\pi f(t_1 + t_2))]$$

$$= 50\, E[\cos(2\pi f(t_1 - t_2))] - 50\, E[\cos(2\pi f(t_1 + t_2))]$$

<u>Aside:</u>

$$E[\cos(2\pi f t)] = \int_0^{100} \frac{\cos(2\pi \alpha t)}{100}\, d\alpha$$

$$= \left[\frac{\sin(2\pi \alpha t)}{200\pi t}\right]_0^{100}$$

$$= \frac{\sin(200\pi t)}{200\pi t} = \operatorname{sinc}(200t)$$

$t \neq 0$, but the formula works if $t = 0$.

$$\Rightarrow R_X(t_1, t_2) = 50\big(\operatorname{sinc}(200(t_1 - t_2)) - \operatorname{sinc}(200(t_1 + t_2))\big)$$

The process is not wide sense stationary because the mean function and the autocorrelation function are not both independent of time origin.

(b) $z = \begin{bmatrix} Y(1\,ms) & Y(2\,ms) \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\Rightarrow Var(z) = \begin{bmatrix} 2 & -1 \end{bmatrix} \times \Lambda_Y \times \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

where

$$\Lambda_Y = \begin{bmatrix} Var(Y(1ms)) & Cov(Y(1ms), Y(2ms)) \\ Cov(Y(1ms), Y(2ms)) & Var(Y(2ms)) \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{L}_Y(1ms, 1ms) & \mathcal{L}_Y(2ms, 1ms) \\ \mathcal{L}_Y(1ms, 2ms) & \mathcal{L}_Y(2ms, 2ms) \end{bmatrix}$$

We find (using the answer of part (a)):

$R_Y(1ms, 1ms) = 50(1 - sinc(0.4)) = 12.159$

$R_Y(2ms, 2ms) = 38.306$

$R_Y(2ms, 1ms) = 21.547$

$m_Y(1ms) = 3.03959$

$m_Y(2ms) = 5.4987$

$$\Rightarrow \begin{cases} \mathcal{L}_Y(1ms, 1ms) = R_Y(1ms, 1ms) - m_Y(1ms)^2 \\ \qquad = 12.159 - 3.0396^2 = 2.9198 \\ \mathcal{L}_Y(2ms, 2ms) = 38.306 - 5.4987^2 = 8.0703 \\ \mathcal{L}_Y(2ms, 1ms) = \mathcal{L}_Y(1ms, 2ms) \\ \qquad = 21.547 - 3.0396 \cdot 5.4987 \\ \qquad = 4.8332 \end{cases}$$

$$\Rightarrow \Lambda_Y = \begin{bmatrix} 2.9198 & 4.8332 \\ 4.8332 & 8.0703 \end{bmatrix}$$

Finally:

$$Var(z) = 2(2 \cdot 2.9198 - 4.8332) - (2 \cdot 4.8332 - 8.0703)$$
$$= 0.4167$$