Solutions #1

The mean and variance of x_i are obtained as follows:

$$\bar{x} = \bar{x}_i = (-1)(1/2) + (0)(1/4) + (1)(1/4)$$
$$= -1/4$$

$$\overline{x^2} = \overline{x_i^2} = (-1)^2(1/2) + (0)^2(1/4) + (1)^2(1/4)$$
$$= 3/4$$

$$\sigma_x^2 = Var(x_i) = 3/4 - (-1/4)^2 = 11/16 = 0.6875$$

(a) From the WLLN we have

$$P(|m - \bar{x}| \geq \varepsilon) \leq \frac{\sigma_x^2}{N\varepsilon^2}$$

$$\Rightarrow \quad P(|m - \underbrace{\bar{x}}_{-1/4}| < \varepsilon) = 1 - P(|m - \bar{x}| \geq \varepsilon)$$
$$\geq \underbrace{1 - \sigma_x^2/N\varepsilon^2}_{95\%} \Rightarrow \varepsilon = \sqrt{\frac{\sigma_x^2}{N(1-0.95)}}$$
$$= 0.11726$$

The range is then given by

$$|m - (-1/4)| < 0.11726$$

Or equivalently

$$\underbrace{-0.11726 - 1/4}_{-0.36726} < m < \underbrace{0.11726 - 1/4}_{-0.13274}$$

We are therefore 95% certain that m lies in the interval $[-0.36726, -0.13274]$.

(b) From the CLT we have

$$P(|m - \underbrace{\overline{X}}_{-1/4}| < \varepsilon) \approx \underbrace{1 - 2Q\left(\frac{\varepsilon\sqrt{N}}{\sigma_X}\right)}_{0.95}$$

$$\Rightarrow Q\left(\varepsilon\sqrt{N/\sigma_X^2}\right) \approx 0.025$$

$$\Rightarrow \varepsilon\sqrt{N/\sigma_X^2} \approx 1.959963$$

$$\Rightarrow \varepsilon \approx \frac{1.959963}{\sqrt{1000/0.6875}} = 0.05139$$

The range is then given by

$$|m - (-1/4)| < 0.05139$$

or equivalently

$$\underbrace{-0.25 - 0.05139}_{-0.30139} < m < \underbrace{0.05139 - 0.25}_{-0.19861}$$

We are therefore 95% certain that m lies in the interval $[-0.30139, -0.19861]$

Solutions #2

(a) From the formulas

$$\underline{m}_{\underline{y}} = \underline{m}_{\underline{x}} \times A^T + \underline{b}$$

$$\Lambda_{\underline{y}} = A \times \Lambda_{\underline{x}} \times A^T$$

where A and $\underline{b}$ are identified in

$$\underline{y} = \underline{x} \times A^T + \underline{b}$$

$$= \underline{x} \times \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} + (0, 1)$$

we immediately obtain

$$\underline{m}_{\underline{y}} = (1, 0) \times \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} + (0, 1)$$

$$= (1, 2)$$

$$\Lambda_{\underline{y}} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} 9 & -4 \\ -4 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 21 & -11 \\ -11 & 21 \end{pmatrix}$$

(b) The hatched region corresponds to

$$0 < \underbrace{2y_1 + 3y_2}_{z} < 24$$

where $z \triangleq 2y_1 + 3y_2 = (y_1, y_2) \times (2, 3)^T$ is Gaussian since $\underline{y}$ is Gaussian. The mean and variance of z are given by the formulas:

$$\bar{z} = \underline{m_y} \times (2, 3)^T$$

$$Var(z) = (2, 3) \times \Lambda_y \times (2, 3)^T$$

and we immediately obtain

$$\bar{z} = (1, 2) \times (2, 3)^T = 8$$

$$Var(z) = (2, 3) \times \begin{pmatrix} 21 & -11 \\ -11 & 21 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$= 141$$

Finally we obtain

$$P(0 \leq z < 24) = Q\left(\frac{0-8}{\sqrt{141}}\right) - Q\left(\frac{24-8}{\sqrt{141}}\right)$$

$$= 0.660837 \ldots$$

$$\boxed{\begin{aligned} &\approx 1 - 0.242 - 0.0888 \\ &= 0.6692 \end{aligned}}$$ with tables

With $\underline{m_y} = (2, 1)$ and $\Lambda_y = \begin{pmatrix} 25 & -10 \\ -10 & 25 \end{pmatrix}$ we would obtain $\bar{z} = 7$, $\sigma_z^2 = 115$. Finally

$$P(0 \leq z < 24) = Q\left(\frac{-7}{\sqrt{115}}\right) - Q\left(\frac{24-7}{\sqrt{115}}\right)$$

$$= 0.6865885 \ldots$$