

Solutions #1

The mean and variance of ν_i are obtained as follows:

$$\bar{\nu} = \bar{\nu}_i = (-1)(1/2) + (0)(1/4) + (1)(1/4) \\ = -1/4$$

$$\bar{\nu}^2 = \bar{\nu}_i^2 = (-1)^2(1/2) + (0)^2(1/4) + (1)^2(1/4) \\ = 3/4$$

$$\sigma_{\nu}^2 = \text{Var}(\nu_i) = 3/4 - (-1/4)^2 = 11/16 = 0.6875$$

(a) From the WLLN we have

$$P(|m - \bar{\nu}| \geq \epsilon) \leq \frac{\sigma_{\nu}^2}{N\epsilon^2} \\ \Rightarrow P(|m - \bar{\nu}| < \epsilon) = 1 - P(|m - \bar{\nu}| \geq \epsilon) \\ \geq 1 - \underbrace{\sigma_{\nu}^2 / N\epsilon^2}_{95\%} \Rightarrow \epsilon = \sqrt{\frac{\sigma_{\nu}^2}{N(1-0.95)}} \\ = 0.11726$$

The range is then given by

$$|m - (-1/4)| < 0.11726$$

Or equivalently

$$\underbrace{-0.11726 - 1/4}_{-0.36726} < m < \underbrace{0.11726 - 1/4}_{-0.13274}$$

We are therefore 95% certain that m lies in the interval $[-0.36726, -0.13274]$.

(b) From the CLT we have

$$P(|m - \underbrace{-1/4}_{\uparrow} | < \varepsilon) \approx \underbrace{1 - 2Q\left(\frac{\varepsilon\sqrt{N}}{\sigma_4}\right)}_{0.95}$$

$$\Rightarrow Q\left(\frac{\varepsilon\sqrt{N}}{\sigma_4}\right) \approx 0.025$$

$$\Rightarrow \frac{\varepsilon\sqrt{N}}{\sigma_4} \approx 1.959963$$

$$\Rightarrow \varepsilon \approx \frac{1.959963}{\sqrt{1000/0.6875}} = 0.05139$$

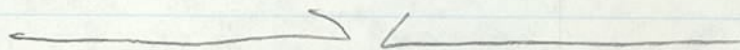
The range is then given by

$$|m - (-1/4)| < 0.05139$$

or equivalently

$$\underbrace{-0.25 - 0.05139}_{-0.30139} < m < \underbrace{0.05139 - 0.25}_{-0.19861}$$

We are therefore 95% certain that m lies in the interval $[-0.30139, -0.19861]$



Solutions #2

(a) From the formulas

$$\underline{m}_y = \underline{m}_x \times A^T + \underline{b}$$

$$\Delta_y = A \times \Delta_x \times A^T$$

where A and \underline{b} are identified in

$$\begin{aligned} \underline{y} &= \underline{x} \times A^T + \underline{b} \\ &= \underline{x} \times \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} + (0, 1) \end{aligned}$$

we immediately obtain

$$\begin{aligned} \underline{m}_y &= (1, 0) \times \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} + (0, 1) \\ &= (1, 2) \end{aligned}$$

$$\begin{aligned} \Delta_y &= \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} 9 & -4 \\ -4 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 21 & -11 \\ -11 & 21 \end{pmatrix} \end{aligned}$$

(b) The hatched region corresponds to

$$0 \leq \underbrace{2y_1 + 3y_2}_z < 24$$

where $z \cong 2y_1 + 3y_2 = (y_1, y_2) \times (2, 3)^T$ is Gaussian since \underline{y} is Gaussian. The mean and variance of z are given by the formulas:

$$\bar{y} = \underline{m}_y \times (2, 3)^T$$

$$\text{Var}(z) = (2, 3) \times \Delta_y \times (2, 3)^T$$

and we immediately obtain

$$\bar{z} = (1, 2) \times (2, 3)^T = 8$$

$$\text{Var}(z) = (2, 3) \times \begin{pmatrix} 21 & -11 \\ -11 & 21 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ = 141$$

Finally we obtain

$$P(0 \leq z < 24) = Q\left(\frac{0-8}{\sqrt{141}}\right) - Q\left(\frac{24-8}{\sqrt{141}}\right) \\ = 0.660837\dots$$

$$\approx 1 - 0.242 - 0.0888 \\ = 0.6692$$

with
tables

With $\underline{m}_y = (2, 1)$ and $\Delta_y = \begin{pmatrix} 25 & -10 \\ -10 & 25 \end{pmatrix}$ we would obtain $\bar{z} = 7$, $\sigma_z^2 = 115$. Finally

$$P(0 \leq z < 24) = Q\left(\frac{-7}{\sqrt{115}}\right) - Q\left(\frac{24-7}{\sqrt{115}}\right) \\ = 0.6865885\dots$$