Solution # 1

1) First calculating $F_y(\beta)$:

$$
\begin{aligned}
F_y(\alpha) &= P(y \le \alpha) \\
&= P(\sqrt{-2\ln(x)} \le \alpha) \\
&= P(-2\ln(x) \le \alpha^2) \\
&= P(\ln(x) \geq -\alpha^2/2) \\
&= P(x \geq e^{-\alpha^2/2}) \\
&= 1 - P(x \le e^{-\alpha^2/2}) \\
&= 1 - F_x(e^{-\alpha^2/2})
\end{aligned}
$$

if $\alpha \geq 0$ otherwise $P(y \le \alpha) = 0$.

since $p_x(\alpha)$ is non-impulsive.

$$
\Rightarrow p_y(\alpha) = \frac{d F_y(\alpha)}{d\alpha} = -p_x\left(e^{-\alpha^2/2}\right) \times e^{-\alpha^2/2} \times (-\alpha)
$$

$$
= \alpha e^{-\alpha^2/2} p_x\left(e^{-\alpha^2/2}\right).
$$

As noted above the latter expression is valide as long as $\alpha \geq 0$; $p_y(\alpha) = 0$, $\forall \alpha < 0$. We also note that for every $\alpha \geq 0$ then $0 \le e^{-\alpha^2/2} \le 1$ and therefore $p_x(e^{-\alpha^2/2}) = 1$, $\forall \alpha \geq 0$. We then obtain

$$
p_y(\alpha) = \alpha e^{-\alpha^2/2} u(\alpha)
$$

and this corresponds to the Rayleigh probability density function of section 2.10.2 in the notes with the parameter $b = 2$.

2) <u>differentiable non-constant function of a random variable</u>

For the transformation $y = g(x) = \sqrt{-2\ln(x)}$ we find:

$$S(\alpha) = \left\{\beta \in \mathbb{R} : \sqrt{-2\ln(\beta)} = \alpha\right\}$$

$$= \begin{cases} \left\{ e^{-\alpha^2/2} \right\} & ; \ \alpha \geq 0 \\ \emptyset & ; \ \alpha < 0 \end{cases}$$

and

$$|g'(\beta)| = \left| \frac{-1}{\beta\sqrt{-2\ln(\beta)}} \right| = \frac{|1/\beta|}{\sqrt{-2\ln(\beta)}}$$

Therefore

$$p_y(\alpha) = \sum_{\beta \in S(\alpha)} \frac{p_x(\beta)}{|g'(\beta)|}$$

$$= \begin{cases} \left|e^{-\alpha^2/2}\right| \sqrt{-2\ln\left(e^{-\alpha^2/2}\right)}\, p_x\left(e^{-\alpha^2/2}\right) & ; \ \alpha \geq 0 \\ 0 & ; \ \alpha < 0 \end{cases}$$

The above is simplified by noticing that $\forall \alpha \geq 0$ we have $0 \leq e^{-\alpha^2/2} \leq 1$ which leads to

$$\underbrace{\left|e^{-\alpha^2/2}\right|}_{e^{-\alpha^2/2}} \underbrace{\sqrt{-2\ln\left(e^{-\alpha^2/2}\right)}}_{\alpha} \underbrace{p_x\left(e^{-\alpha^2/2}\right)}_{1} = \alpha e^{-\alpha^2/2}$$

Therefore $p_y(\alpha) = \alpha e^{-\alpha^2/2} u(\alpha)$. This is the Rayleigh probability density function of section 2.10.2 in the notes with the parameter $b = 2$.

3) Reversible transformation of random variables:

The transformation

$$y = f(x) = \sqrt{-2\ln(x)}$$

is reversible as a function

$$f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$$

The inverse is

$$g: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$$

$$g: \beta \mapsto e^{-\beta^2/2}$$

We may then write for every $\beta \geq 0$:

$$p_y(\beta) = p_x(g(\beta)) \; |J_g(\beta)|$$

where $p_x(g(\beta)) = p_x(e^{-\beta^2/2}) = 1$, $\forall \beta \geq 0$ (because $0 \leq e^{-\beta^2/2} \leq 1$, $\forall \beta \geq 0$) and

$$|J_g(\beta)| = |g'(\beta)| = |-\beta e^{-\beta^2/2}| \underset{\forall \beta \geq 0}{=} \beta e^{-\beta^2/2}$$

It follows that:

$$p_y(\beta) = \begin{cases} \beta e^{-\beta^2/2} & ; \text{ if } \beta \geq 0 \\ 0 & ; \text{ elsewhere} \end{cases}$$

This is the Rayleigh probability density function of section 2.10.2 in the notes with the parameter $b = 2$.

Solution # 2:

Available from the course web page.

Solution # 3

(a) $P(A) = \int_{-\infty}^{\infty} f_X(\alpha, A)\, d\alpha$

$= \int_{0}^{\infty} \frac{1}{4} e^{-\alpha}\, d\alpha$

$= \frac{1}{4} \left[-e^{-\alpha} \right]_0^{\infty} = 1/4$

$P(B) = \int_{-\infty}^{\infty} f_X(\alpha, B)\, d\alpha$

$= \int_0^1 1/2\, d\alpha = 1/2$

$P(A \cup B) = P(A) + P(B) = 1/4 + 1/2 = 3/4$

cuz $A \cap B = \emptyset$

(b) $P(\{X \geq 1/2\} | A) = \int_{1/2}^{\infty} f_X(\alpha | A)\, d\alpha$

$= \int_{1/2}^{\infty} \frac{f_X(\alpha, A)}{P(A)}\, d\alpha$

$= 4 \int_{1/2}^{\infty} \frac{1}{4} e^{-\alpha}\, d\alpha$

$= \left[-e^{-\alpha} \right]_{1/2}^{\infty} = e^{-1/2} = 1/\sqrt{e}$

(c) $P(A | X = 1/2) = \frac{f_X(1/2, A)}{f_X(1/2)} = \frac{16}{11 + 4e^{-1/2}} \times \frac{e^{-1/2}}{4} = \frac{4}{11\sqrt{e} + 4}$

$\approx 0.1807017.$