Solutions

$$s = \begin{cases} \sqrt{E} \triangleq s_0 & \text{for } m_0 \\ -\sqrt{E} \triangleq s_1 = -s_0 & \text{for } m_1 \end{cases}$$

1. $p_{r_1 r_2}(\rho_1, \rho_2, \{s = s_\ell\})$

$$= \frac{1}{2} \frac{1}{2\pi\sigma^2} \exp\left(\frac{-(\rho_1 - s_\ell)^2 - (\rho_2 - \rho_1 + s_\ell)^2}{2\sigma^2} \right)$$

$$= \frac{1}{4\pi\sigma^2} \exp\left(\frac{-\rho_1^2 - s_\ell^2 + 2\rho_1 s_\ell - \rho_2^2 - \rho_1^2 - s_\ell^2 + 2\rho_1\rho_2 - 2\rho_2 s_\ell + 2\rho_1 s_\ell}{2\sigma^2} \right)$$

and $\hat{m}(\rho_1, \rho_2) = m_0$ iff

$$p_{r_1 r_2}(\rho_1, \rho_2, \{s = s_0\}) > p_{r_1 r_2}(\rho_1, \rho_2, \{s = s_1\})$$

$$\Updownarrow$$

notice that $s_0^2 = s_1^2$

$$\cancel{-2\rho_1^2} \cancel{-2s_0^2} + 4\rho_1 s_0 \cancel{- \rho_2^2} \cancel{+ 2\rho_1\rho_2} - 2\rho_2 s_0 > \cancel{-2\rho_1^2} \cancel{-2s_1^2} + 4\rho_1 s_1 \cancel{- \rho_2^2} \cancel{+ 2\rho_1\rho_2} - 2\rho_2 s_1$$

$$\Updownarrow$$

$$4\rho_1(s_0 - s_1) - 2\rho_2(s_0 - s_1) > 0$$

$$\Updownarrow$$

$$2\rho_1 - \rho_2 > 0$$

The decision device $\hat{m}(\ )$ is then

$$\hat{m}(\rho_1, \rho_2) \begin{cases} m_0; & \text{if } 2\rho_1 - \rho_2 > 0 \\ m_1; & \text{elsewhere} \end{cases}$$

2. As stated $P(\mathcal{E}) = P(\mathcal{E} \mid \{s = s_1\})$ and from the above decision device we have

$$P(\mathcal{E} \mid \{s = s_1\}) = P(\{2r_1 - r_2 > 0\} \mid \{s = s_1\})$$

$$= P(\{2(s_1 + n_1) - (n_1 + n_2) > 0\} \mid \{s = s_1\})$$

$$= P(\{n_1 - n_2 > -2s_1\} \mid \{s = s_1\})$$

$$= P(\{n > 2\sqrt{E}\} \mid \{s = s_1\})$$

where we define $n = n_1 - n_2$. Because s, n_1, n_2 are jointly statistically independent and n is function of n_1, n_2 only, n and s are statistically independent. In addition, n is a Gaussian random variable (refer to Theorem 19 in the notes). Therefore

$$P(\mathcal{E}) = P(\mathcal{E} \mid \{s = s_1\}) = P(\{n > 2\sqrt{E}\})$$

$$= Q\left(\frac{2\sqrt{E} - \overline{n}}{\sqrt{\mathrm{Var}(n)}}\right)$$

We easily find:

$$\overline{n} = \overline{n_1} - \overline{n_2} = 0 - 0 = 0$$

$$\begin{aligned} \mathrm{Var}(n) &= \mathrm{Var}(n_1 - n_2) \\ &= \mathrm{Var}(n_1) + \mathrm{Var}(n_2) \\ &= 2\sigma^2 \end{aligned}$$

Finally,

$$P(\mathcal{E}) = Q\left(\sqrt{\frac{2E}{\sigma^2}}\right)$$