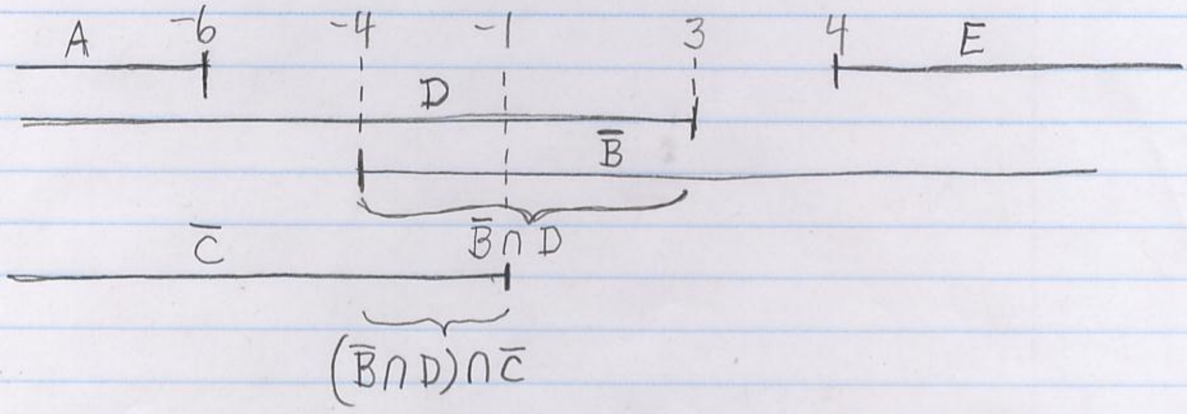
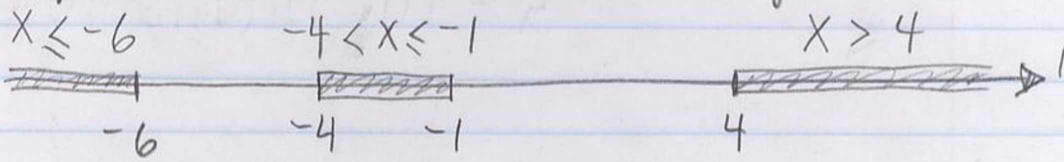
Solution # 1



On voit immédiatement que $A \cup ((\overline{B} \cap D) \cap \overline{C}) \cup E$ correspond au trois intervals disjoint :



La variable X est Gaussienne de moyenne 2 et variance 16. On obtient alors :

$$P(X \leq -6) = Q\left(\frac{2+6}{4}\right) = Q(2) = 0.02275$$

$$P(-4 < X \leq -1) = Q\left(\frac{-4-2}{4}\right) - Q\left(\frac{-1-2}{4}\right)$$
$$= 1 - Q(1.5) - (1 - Q(0.75)) = 0.159820$$

$$P(X > 4) = Q\left(\frac{4-2}{4}\right) = Q(1/2) = 0.30854$$

$$\Rightarrow P(A \cup ((\overline{B} \cap D) \cap \overline{C}) \cup E) = 0.02275 + 0.15982 + 0.30854$$
$$= 0.491108$$

Solution # 2

1.

$$\begin{aligned}
P(r_0\,|\,m_0)P(m_0) &= 0.4 \times 0.4 = 0.16 \\
P(r_0\,|\,m_1)P(m_1) &= 0.6 \times 0.3 = 0.18 \;\leftarrow\; \text{largest} \\
P(r_0\,|\,m_2)P(m_2) &= 0.5 \times 0.3 = 0.15
\end{aligned}$$

$$\begin{aligned}
P(r_1\,|\,m_0)P(m_0) &= 0.6 \times 0.4 = 0.24 \;\leftarrow\; \text{largest} \\
P(r_1\,|\,m_1)P(m_1) &= 0.4 \times 0.3 = 0.12 \\
P(r_1\,|\,m_2)P(m_2) &= 0.5 \times 0.3 = 0.15
\end{aligned}$$

The optimal decision rule is then:

$$\begin{array}{lccc}
\widehat{m}_{\text{opt}}: & \{r_0,\, r_1\} & \rightarrow & \{m_0,\, m_1,\, m_2\} \\
\widehat{m}_{\text{opt}}: & r_0 & \mapsto & m_1 \\
\widehat{m}_{\text{opt}}: & r_1 & \mapsto & m_0
\end{array}$$

2. In light of the previous question, $\widehat{m}_G(\,)$ is not optimal. Clearly the decision is m_2 iff r_0 is received. We then have:

$$\begin{aligned}
P(\widehat{m}_G(r_i) = m_2) &= P(r_0), \text{ i.e. probability that } r_0 \text{ is received} \\
&= P(m_0,\, r_0) + P(m_1,\, r_0) + P(m_2,\, r_0) \\
&= P(r_0\,|\,m_0)P(m_0) + P(r_0\,|\,m_1)P(m_1) + P(r_0\,|\,m_2)P(m_2) \\
&= 0.4 \times 0.4 + 0.3 \times 0.6 + 0.3 \times 0.5 \\
&= 0.49
\end{aligned}$$

3. $P(\text{error}) = 1 - P(\text{correct})$ where $P(\text{correct})$ is given by:

$$\begin{aligned}
P(\text{correct}) &= P(\widehat{m}_G(r_0),\, r_0) + P(\widehat{m}_G(r_1),\, r_1) \\
&= P(m_2,\, r_0) + P(m_0,\, r_1) \\
&= P(r_0\,|\,m_2)P(m_2) + P(r_1\,|\,m_0)P(m_0) \\
&= 0.5 \times 0.3 + 0.6 \times 0.4 \\
&= 0.39
\end{aligned}$$

It then follows that $P(\text{error}) = 0.61$.

Remark. $P(\text{error})$ with the optimal decision rule $\widehat{m}_{\text{opt}}(\,)$ found in question 1 would be:

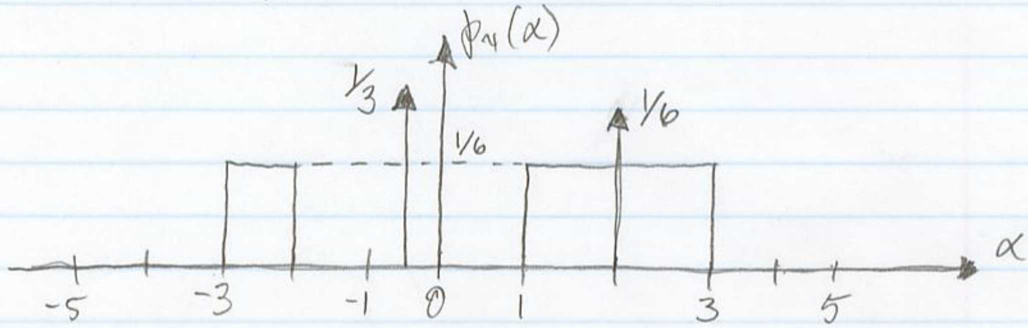
$$\begin{aligned}
P(\text{correct}) &= P(m_1,\, r_0) + P(m_0,\, r_1) \\
&= 0.6 \times 0.3 + 0.6 \times 0.4 \\
&= 0.42
\end{aligned}$$

and we obtain $P(\text{error}) = 0.58$, i.e. slightly smaller than that of the receiver using decision rule $\widehat{m}_G(\,)$.

Solution # 3

$$p_X(\alpha) = \frac{d F_X(\alpha)}{d\alpha}$$

We easily find:



Check: the total area is $1 \cdot 1/6 + 1/3 + 2 \cdot 1/6 + 1/6 = 1$. O.K.

This is the probability density function sketched in figure 2.7 of the notes.