Solution # 1:

(a) $P(A) = P(B) = P(C) = 1/2$

$A \cap B = \{2, 3\}$ $\quad A \cap C = \{3, 4\}$ $\quad B \cap C = \{3, 6\}$

$\Rightarrow \underbrace{P(A \cap B) = P(A)P(B)}_{= 1/4}, \underbrace{P(A \cap C) = P(A)P(C)}_{= 1/4}, \underbrace{P(B \cap C) = P(B)P(C)}_{= 1/4}$

Finally $A \cap B \cap C = \{3\}$ and $P(A \cap B \cap C) = P(A)P(B)P(C) = 1/8$

A, B, C are statistically independent.

(b) E and F are statistically independent by theorem 3 at page 10 in the notes.

Solution # 2

First have to find the best decision rule:

1) if r_1 is received then

$$P(m_0)\,P(r_1|m_0) = 1/3 \cdot 1/2 = 1/6$$
$$P(m_1)\,P(r_1|m_1) = 0$$
$$P(m_2)\,P(r_1|m_2) = 1/3 \cdot 1 = 1/3 \longleftarrow \text{largest.}$$

So if r_1 is received we guess m_2.

2) if r_2 is received then

$$P(m_0)\,P(r_2|m_0) = 1/3 \cdot 1/2 = 1/6$$
$$P(m_1)\,P(r_2|m_1) = 1/3 \cdot 1 = 1/3 \longleftarrow \text{largest.}$$
$$P(m_2)\,P(r_2|m_2) = 0$$

So if r_2 is received we guess m_1.

The decision rule is then

$$r_1 \longmapsto m_2$$
$$r_2 \longmapsto m_1$$

and we never guess m_0.

Next we find $P(e)$:

$$P(e) = P(m_2, r_1) + P(m_1, r_2)$$

transmit

received

$$= P(m_2)\,P(r_1|m_2) + P(m_1)\,P(r_2|m_1)$$
$$= 1/3 \cdot 1 + 1/3 \cdot 1 = 2/3 \Rightarrow P(\varepsilon) = 1 - P(e)$$
$$= 1/3$$

Solution # 3

Consider the probability space defined by:[1]

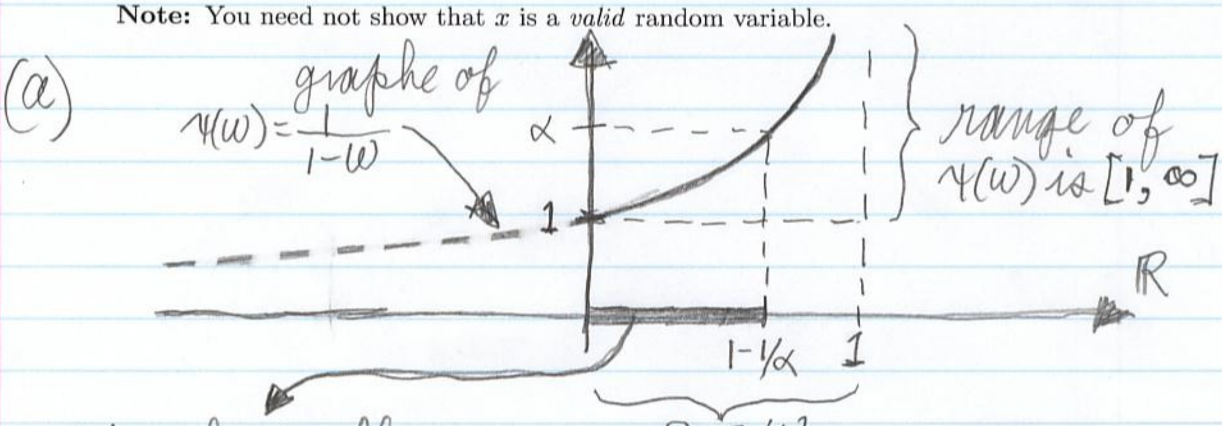
- the sample space is the real-line interval $\Omega = \{\omega : 0 \le \omega \le 1\}$;
- the class of events is the set of all intervals of this line segment, plus (countable) unions, (countable) intersections, and complements of such intervals. The intervals may include both, one, or either of the end points;
- the probability assignment is the sum of the lengths of the disjoint intervals that constitute the event.

Calculate the probability distribution function $F_x(\alpha)$ of the random variable x defined by:

$$\begin{array}{rccc} x: & \Omega & \rightarrow & \mathbb{R} \\ x: & \omega & \mapsto & \frac{1}{1-\omega} \end{array}$$

Note: You need not show that x is a *valid* random variable.

(a)



interval of all
$\omega : x(\omega) \le \alpha$ → its length is $1 - 1/\alpha$

$$\Rightarrow F_x(\alpha) = \begin{cases} 1 - 1/\alpha \ ; & \text{if } \alpha > 1 \\ 0 \ ; & \text{if } \alpha \le 1 \end{cases}$$

[1] W&J, bottom of page 21 and top of page 22.

Analytically: Since $\psi(\omega) \in [1, \infty]$ we have:

$$F_\psi(\alpha) \triangleq P(\psi(\omega) \leq \alpha)$$
$$= \begin{cases} 0 & ; \text{ if } \alpha \leq 1 \\ P\left(\frac{1}{1-\omega} \leq \alpha\right) & ; \text{ if } \alpha > 1 \end{cases}$$

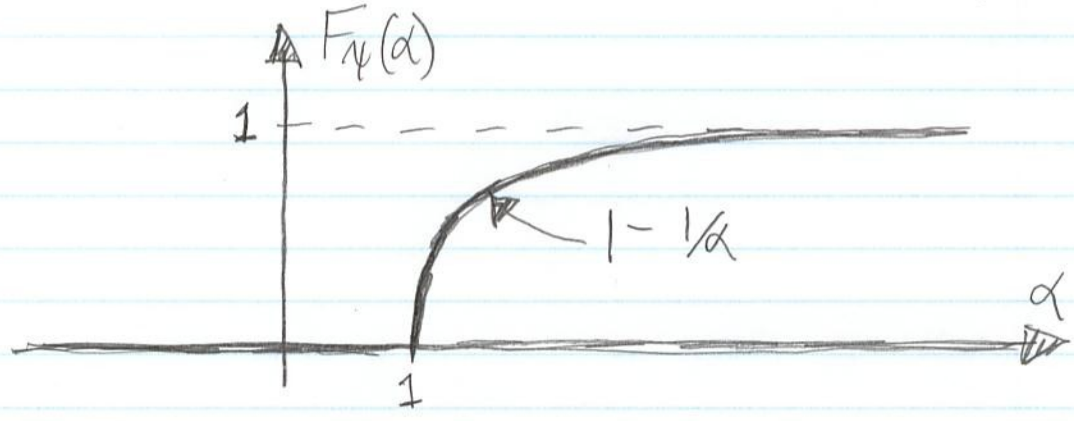
When $\alpha > 1$ we have

$$\frac{1}{1-\omega} \leq \alpha \iff 1 \leq \alpha(1-\omega) = \alpha - \alpha\omega$$
$$\iff \alpha\omega \leq \alpha - 1$$
$$\iff \omega \leq \frac{\alpha-1}{\alpha} = 1 - \frac{1}{\alpha}$$

since $1-\omega > 0$

since $\alpha > 0$

Therefore $P\left(\frac{1}{1-\omega} \leq \alpha\right) = P(\omega \leq 1 - 1/\alpha)$
$= P(\omega \in [0, 1-1/\alpha])$
$= 1 - 1/\alpha$ (length of the interval)



(b) $P(\{\omega : 2 < \psi(\omega) \leq 3\}) = F_\psi(3) - F_\psi(2)$
$= (1 - 1/3) - (1 - 1/2)$
$= 1/2 - 1/3 = 1/6 \approx 0.16667.$