Solution problem 2.25 W&J

(a) Suppose $r_1 = \rho_1$, $r_2 = \rho_2$ is received. The decision rule is:

$$(\rho_1,\, \rho_2) \leftrightarrow m_0 \quad \Leftrightarrow \quad P(m_0 \,|\, r_1 = \rho_1,\, r_2 = \rho_2) \geq P(m_1 \,|\, r_1 = \rho_1,\, r_2 = \rho_2)$$

$$\Leftrightarrow \quad \frac{p_{r_1 r_2}(\rho_1,\, \rho_2 \,|\, m_0) P(m_0)}{p_{r_1 r_2}(\rho_1,\, \rho_2)} \geq \frac{p_{r_1 r_2}(\rho_1,\, \rho_2 \,|\, m_1) P(m_1)}{p_{r_1 r_2}(\rho_1,\, \rho_2)}$$

$$\Leftrightarrow \quad p_{r_1 r_2}(\rho_1,\, \rho_2 \,|\, m_0) P(m_0) \geq p_{r_1 r_2}(\rho_1,\, \rho_2 \,|\, m_1) P(m_1)$$

But

$$\begin{aligned} p_{r_1 r_2}(\rho_1,\, \rho_2 \,|\, m_0) &= p_{n_1 n_2}(\rho_1 - s_0,\, \rho_2 - s_0 \,|\, m_0) \\ &= p_{n_1}(\rho_1 - s_0) p_{n_2}(\rho_2 - s_0) \end{aligned}$$

since $r_1 = s_0 + n_1$, $r_2 = s_0 + n_2$ when m_0 is the message (i.e. s_0 is transmitted) and $\{m_0\}$, n_1, n_2 are all independent. Similarly we have:

$$p_{r_1 r_2}(\rho_1,\, \rho_2 \,|\, m_1) = p_{n_1}(\rho_1 - s_1) p_{n_2}(\rho_2 - s_1)$$

The decision rule becomes:

$$(\rho_1,\, \rho_2) \leftrightarrow m_0 \quad \Leftrightarrow \quad p_{n_1}(\rho_1 - s_0) p_{n_2}(\rho_2 - s_0) P(m_0) \geq p_{n_1}(\rho_1 - s_1) p_{n_2}(\rho_2 - s_1) P(m_1)$$

$$\Leftrightarrow \quad \frac{e^{-(\rho_1 - s_0)^2/(2\sigma_1^2)}}{\sqrt{2\pi\sigma_1^2}} \frac{e^{-(\rho_2 - s_0)^2/(2\sigma_2^2)}}{\sqrt{2\pi\sigma_2^2}} P(m_0) \geq P(m_1) \frac{e^{-(\rho_1 - s_1)^2/(2\sigma_1^2)}}{\sqrt{2\pi\sigma_1^2}} \frac{e^{-(\rho_2 - s_1)^2/(2\sigma_2^2)}}{\sqrt{2\pi\sigma_2^2}}$$

$$\Leftrightarrow \quad \ln(P(m_0)) - \frac{(\rho_1 - s_0)^2}{2\sigma_1^2} - \frac{(\rho_2 - s_0)^2}{2\sigma_2^2} \geq \ln(P(m_1)) - \frac{(\rho_1 - s_1)^2}{2\sigma_1^2} - \frac{(\rho_2 - s_1)^2}{2\sigma_2^2}$$

$$\Leftrightarrow \quad \frac{\rho_1}{\sigma_1^2} + \frac{\rho_2}{\sigma_2^2} \geq \frac{1}{2\sqrt{E}} \ln\left(\frac{P(m_1)}{P(m_0)}\right)$$

where we have used $s_0 = \sqrt{E}$, $s_1 = -\sqrt{E}$. The decision regions are shown in figure 1. Calculating the probability of error on the decision regions of figure 1 would require integration of a bivariate joint Gaussian density function. Instead we see that the receiver can be simplified by implementing it as shown in figure 2 where the decision is based on the random variable $z \triangleq \frac{r_1}{\sigma_1^2} + \frac{r_2}{\sigma_2^2}$. The decision based on $z = \gamma$ is as follows:

$$\gamma = \frac{\rho_1}{\sigma_1^2} + \frac{\rho_2}{\sigma_2^2} \leftrightarrow m_0 \Leftrightarrow \gamma \geq \underbrace{\frac{1}{2\sqrt{E}} \ln\left(\frac{P(m_1)}{P(m_0)}\right)}_{\text{threshold} \equiv a}$$

and the decision regions on z are shown in figure 3. Since r_1, r_2 are Gaussian and independent when conditioned on m_0 or m_1, then z is Gaussian:

$$p_z(\gamma \,|\, \{m_i\}) = \frac{1}{\sqrt{2\pi\sigma_z^2}} e^{-(\gamma - \bar{z})^2/(2\sigma_z^2)}$$

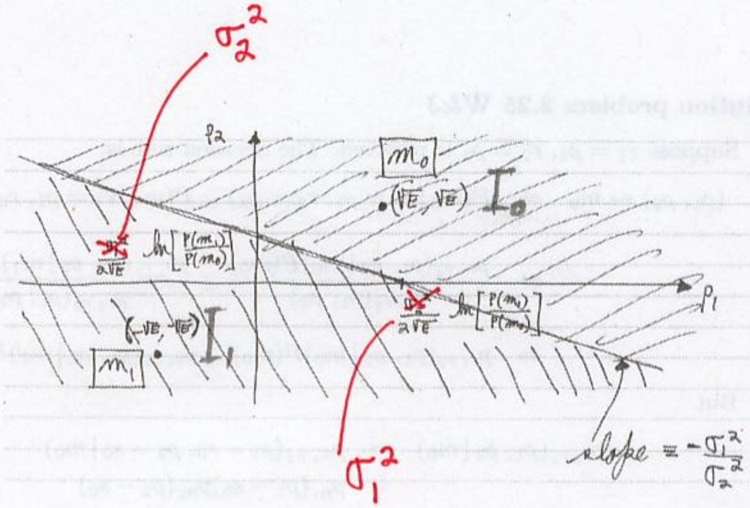


Figure 1:

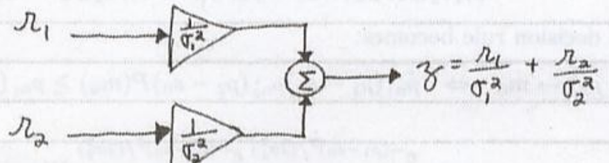


Figure 2:

Its mean $\overline{z}$ and variance σ_z^2 are respectively given by:

$$
\begin{aligned}
\overline{z} \triangleq E[z] &= \frac{E[r_1]}{\sigma_1^2} + \frac{E[r_2]}{\sigma_2^2} \\
&= \begin{cases} \frac{E[\sqrt{E}+n_1]}{\sigma_1^2} + \frac{E[\sqrt{E}+n_2]}{\sigma_2^2} = \frac{\sqrt{E}}{\sigma_1^2} + \frac{\sqrt{E}}{\sigma_2^2} & ; \text{ if } m_0 \text{ is the message} \\ \\ \frac{E[-\sqrt{E}+n_1]}{\sigma_1^2} + \frac{-E[\sqrt{E}+n_2]}{\sigma_2^2} = \frac{-\sqrt{E}}{\sigma_1^2} + \frac{-\sqrt{E}}{\sigma_2^2} & ; \text{ if } m_1 \text{ is the message} \end{cases} \\
\sigma_z^2 \triangleq Var[z] &= \frac{1}{\sigma_1^4} Var[r_1] + \frac{1}{\sigma_2^4} Var[r_2] \text{ for both messages } m_0 \text{ and } m_1 \\
&= \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \text{ for both messages } m_0 \text{ and } m_1
\end{aligned}
$$

Clearly

$$
P(\mathscr{E}) = P(\mathscr{E} \mid m_0)P(m_0) + P(\mathscr{E} \mid m_1)P(m_1) \tag{1}
$$

where

$$
P(\mathscr{E} \mid m_0) = \int_{I_1} p_z(\gamma \mid m_0) \mathrm{d}\gamma
$$

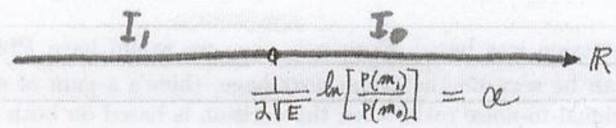


Figure 3:

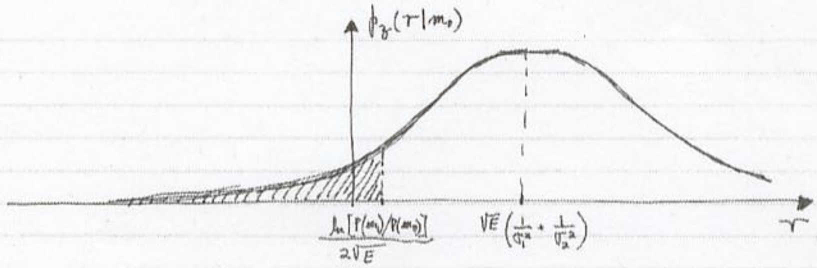


Figure 4:

The probability of error given the message m_0, is shown as the shaded area in figure 4. It is given by:

$$Q\left(\frac{\sqrt{E}(1/\sigma_1^2 + 1/\sigma_2^2) - a}{\sigma_z}\right) \tag{2}$$

where we recall $a = \frac{1}{2\sqrt{E}} \ln\left(\frac{P(m_1)}{P(m_0)}\right)$. Similarly we would find

$$P(\mathscr{E}\,|\,m_1) = Q\left(\frac{a + \sqrt{E}(1/\sigma_1^2 + 1/\sigma_2^2)}{\sigma_z}\right) \tag{3}$$

since $E[z] = -\sqrt{E}(1/\sigma_1^2 + 1/\sigma_2^2)$ when the message is m_1. Finally $P(\mathscr{E})$ is obtained by replacing equations (2) and (3) in equation (1).

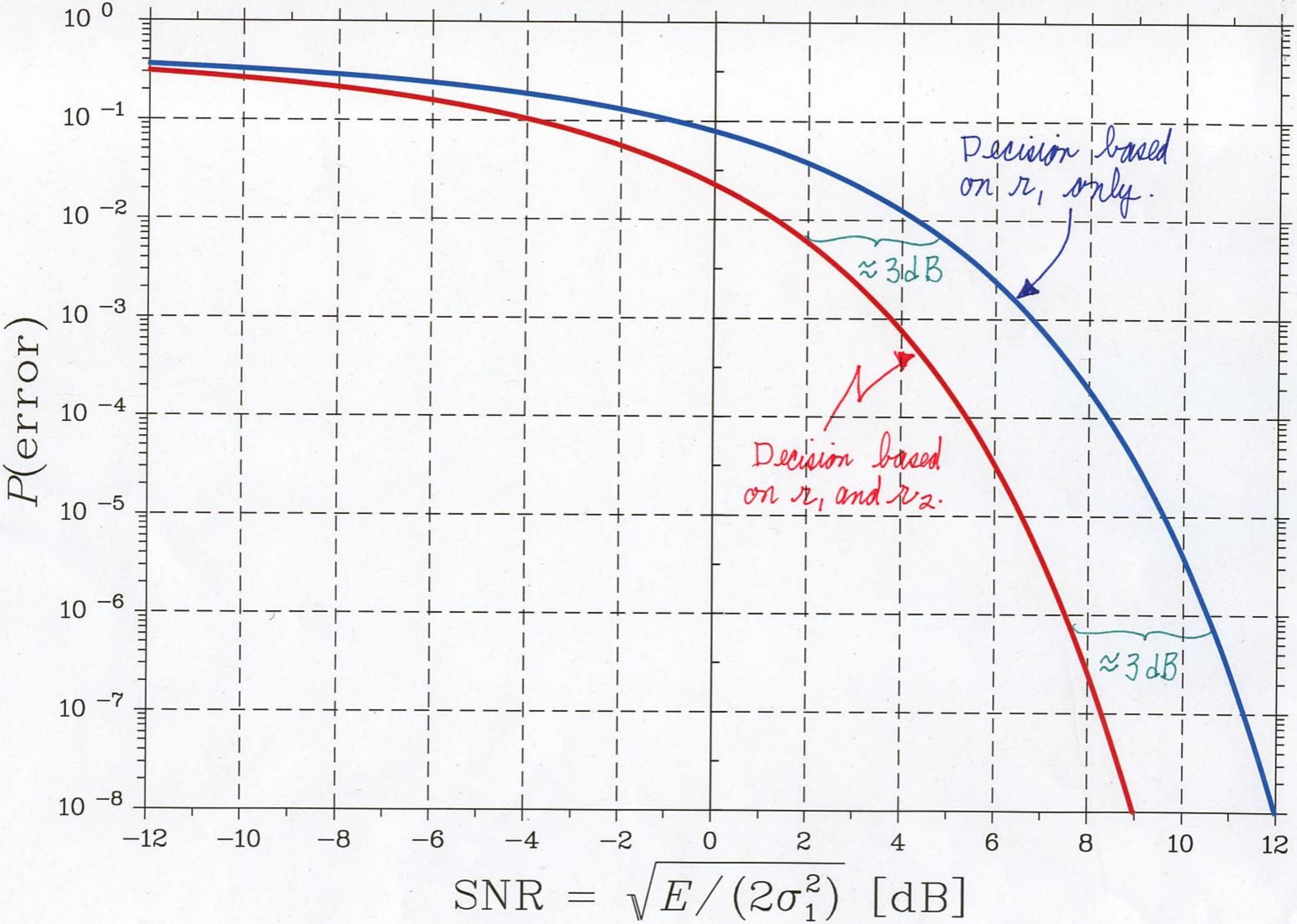
(b) If $\sigma_1^2 = \sigma_2^2$, $P(m_0) = P(m_1)$ then the optimum decision rule in terms of $z = \gamma$ simplifies to:

$$\gamma \triangleq \frac{\rho_1 + \rho_2}{\sigma_1^2} \leftrightarrow m_0 \Leftrightarrow \gamma \geq \underbrace{0 = a}_{\text{threshold}}$$

It follows that:

$$P(\mathscr{E}\,|\,m_0) = Q\left(\sqrt{2E/\sigma_1^2}\right) = Q\left(\frac{\sqrt{2E}}{\sigma_1}\right) = P(\mathscr{E})$$

If detection was based on r_1 only then we would have $P(\mathscr{E}) = Q\left(\frac{\sqrt{E}}{\sigma_1}\right)$. As can be seen on the graph next page, there's a gain of about 3 dB on the signal-to-noise ratio when the decision is based on both r_1 and r_2 (red curve) over the decision based on r_1 only (blue curve).



10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
$P(\mathrm{error})$
−12
−10
−8
−6
−4
−2
0
2
4
6
8
10
12
$\mathrm{SNR} = \sqrt{E/(2\sigma_1^2)}$ [dB]
Decision based on r_1 only.
Decision based on r_1 and r_2.
≈ 3dB
≈ 3 dB