Name: ______________________ College Number: ______________________

EE501: An Introduction to the Theory of Statistical Communications

Tuesday, 24 November 2015

Fourth Quiz

REMARKS:

1. Hand held calculator is allowed,
2. Open book quiz,
3. Formula sheets are attached,
4. Marks distribution:
   Question #1: 3 points
   Question #2: 4 points
   Question #3: 3 points
5. Justify all your answers.

# 1	
# 2	
# 3	

1. Consider the random process defined by $x(t) = 10\sin(2\pi f t)$ where f is a random variable uniformly distributed between 0 Hz and 100 Hz. This is very similar to the random process of equations 3.24(a) and 3.24(b) on page 143 in Wozencraft & Jacobs. The process is not stationary.

   (a) Calculate the mean function $m_x(t)$ and the autocorrelation function $\mathscr{R}_x(t_1, t_2)$ of $x(t)$ and show that:

   $$\begin{aligned} m_x(t) &= \frac{1-\cos(200\pi t)}{20\pi t} \\ \mathscr{R}_x(t_1,t_2) &= 50\Big(\text{sinc}\big(200(t_1-t_2)\big) - \text{sinc}\big(200(t_1+t_2)\big)\Big) \end{aligned}$$

   Is the process wide sense stationary? Justify your answer.

   (b) Calculate $\text{Cov}\big(x(1\text{ ms}), x(2\text{ ms})\big)$, the covariance between random variables $x(1\text{ ms})$ and $x(2\text{ ms})$.

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2. Let $x(t)$ denote the stationary Binary Random Process of example 3.3.5. With a rectangular pulse of duration $T = 1$ ms. We have found that

$$m_x(t) = 0$$

$$\mathscr{R}_x(\tau) = \begin{cases} 1 - 1000\,|\tau| & ; \quad -1 \text{ ms} < \tau < 1 \text{ ms} \\ \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

$$S_x(f) = 0.001\ \text{sinc}^2(f/1000).$$

$S_x(f)$ was also sketched in figure 3.12. $x(t)$ is fed through 2 filters as shown in figure 1. The filters frequency responses are given by:

$$H_y(f) = \begin{cases} 1 & ; \quad -2 \text{ kHz} < f < 2 \text{ kHz} \\ \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

$$H_z(f) = \begin{cases} 1 & ; \quad |f| > 1 \text{ kHz} \\ \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

$x(t)$, $y(t)$, $z(t)$ are all jointly stationary non-Gaussian random processes.

(a) Calculate and/or sketch $S_y(f)$, $S_z(f)$, $S_{yz}(f)$.

(b) Calculate the total average power of $y(t)$ and $z(t)$.
Hint: You need not calculate $\mathscr{R}_y(\tau)$, $\mathscr{R}_z(\tau)$.

(c) Are the processes $y(t)$, $z(t)$ statistically independent? Justify your answer.

Note: The following may be useful ($a > 0$):

$$\int_{-a}^{a} \text{sinc}^2(x)\text{d}x = \frac{\cos(2\pi a) - 1 + 2\pi a\,\text{Si}(2\pi a)}{\pi^2 a}$$

$$\int_{|x|>a} \text{sinc}^2(x)\text{d}x = \frac{\pi^2 a + 1 - \cos(2\pi a) - 2\pi a\,\text{Si}(2\pi a)}{\pi^2 a}$$

and a table of the function Si( ) is attached.

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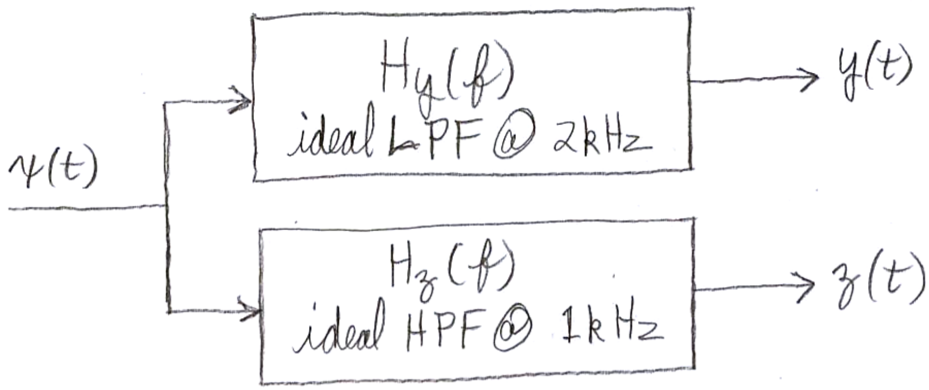

Figure 1:

3. Consider the signals $s_1(t)$, $s_2(t)$ in figure 2.

   (a) Calculate an orthonormal basis for $s_1(t)$, $s_2(t)$.
   Hint: you will find two orthonormal functions $\phi_1(t)$ and $\phi_2(t)$.

   (b) Calculate the coordinates of $s_1(t)$ and $s_2(t)$ in the basis $\{\phi_1(t),\ \phi_2(t)\}$.

   (c) Calculate the distance between $s_1(t)$, $s_2(t)$.

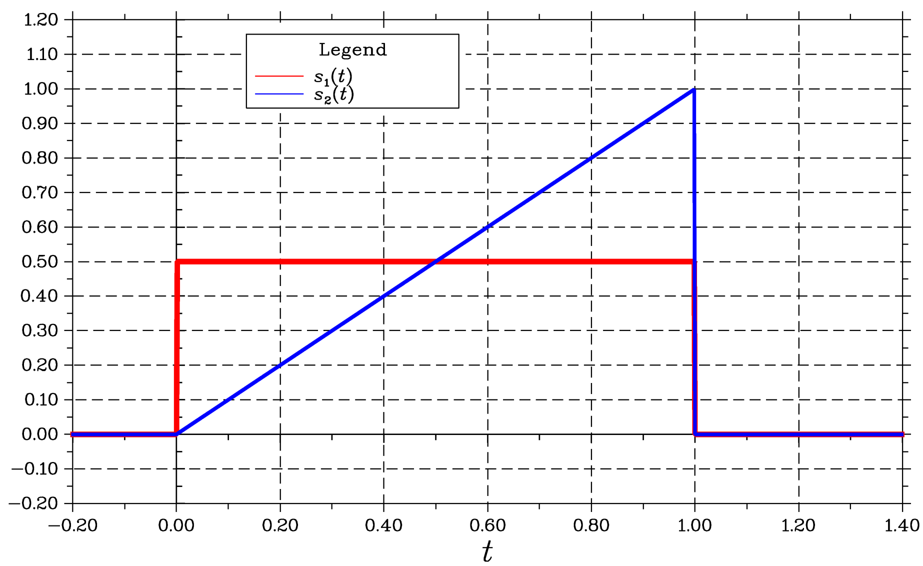

Figure 2:

END

Formula Sheets

$\cos\theta = \sin(\theta + 90°)$

$2\cos u \cos v = \cos(u-v) + \cos(u+v)$

$2\sin u \sin v = \cos(u-v) - \cos(u+v)$

$2\sin u \cos v = \sin(u-v) + \sin(u+v)$

$2\sin\theta\cos\theta = \sin(2\theta)$

$2\cos^2\theta = 1 + \cos(2\theta)$

$2\sin^2\theta = 1 - \cos(2\theta)$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

$P(AB) = P(A|B)P(B) = P(B|A)P(A)$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$p_x(\alpha) = \frac{\mathrm{d}F_x(\alpha)}{\mathrm{d}\alpha}$

$F_x(\alpha) = P(\{x \le \alpha\}) = \int_{-\infty}^{\alpha} p_x(u)\,\mathrm{d}u$

$P(a < x \le b) = \int_a^b p_x(\alpha)\mathrm{d}\alpha$

$p_x(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\alpha-\mu)^2/(2\sigma^2)}$

$$\begin{aligned} Q(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_\alpha^\infty e^{-\beta^2/2}\mathrm{d}\beta \\ &= \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{\alpha}{\sqrt{2}}\right)\right] \\ &= 1 - Q(-\alpha) \end{aligned}$$

$$\begin{aligned} \mathrm{erf}(\alpha) &= \frac{2}{\sqrt{\pi}} \int_0^\alpha e^{-\beta^2}\mathrm{d}\beta \\ &= 1 - 2Q(\sqrt{2}\alpha) \end{aligned}$$

$P(x > a) = Q\left(\frac{a-\mu}{\sigma}\right)$

$P(a < x \le b) = Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right)$

$P(x \le a) = Q(\frac{\mu-a}{\sigma})$

$\frac{\mathrm{d}Q(\alpha)}{\mathrm{d}\alpha} = \frac{-1}{\sqrt{2\pi}} e^{-\alpha^2/2}$

$p_x(\alpha) = \int_{-\infty}^{\infty} p_{xy}(\alpha,\beta)\mathrm{d}\beta$

$y = bx + a \Rightarrow p_y(\alpha) = \frac{1}{|b|} p_x(\frac{\alpha-a}{b})$

$$p_{g(x)}(\beta) = p_y(\beta) = \begin{cases} \sum\limits_{\alpha \in S(\beta)} \frac{p_x(\alpha)}{|g'(\alpha)|} & ; \text{ if } S(\beta) \ne \emptyset \text{ and} \\ & \quad g'(\alpha) \ne 0, \forall \alpha \in S(\beta) \triangleq \{\alpha \in \mathbb{R} : \beta = g(\alpha)\} \\ 0 & ; \text{ if } S(y) = \emptyset \end{cases}$$

$p_{\boldsymbol{f}(\boldsymbol{x})}(\boldsymbol{\beta}) = p_{\boldsymbol{y}}(\boldsymbol{\beta}) = p_{\boldsymbol{x}}(\boldsymbol{g}(\boldsymbol{\beta}))|J_{\boldsymbol{g}}(\boldsymbol{\beta})|$

$p_x(\alpha|y=v) = p_{x|y}(\alpha, v) = \frac{p_{xy}(\alpha,v)}{p_y(v)}$

x, y are independent $\Leftrightarrow p_{xy}(\alpha,\beta) = p_x(\alpha)p_y(\beta)$

Formula Sheets (continued)

Fourier Transform Properties

Operation	$g(t)$	$G(f)$
Addition	$g_1(t) + g_2(t)$	$G_1(f) + G_2(f)$
Multiplication by a constant	$ag(t)$	$aG(f)$
Symmetry	$G(t)$	$g(-f)$
Scaling	$g(at)$	$\frac{1}{\lvert a\rvert}G(\frac{f}{a})$
Time shifting	$g(t - t_0)$	$e^{-j2\pi f t_0}G(f)$
Frequency Shifting	$e^{j2\pi f_0 t}g(t)$	$G(f - f_0)$
Modulation	$2g(t)\cos(2\pi f_c t)$	$G(f - f_c) + G(f + f_c)$
Time Differentiation	$\frac{\mathrm{d}^k g(t)}{\mathrm{d}t^k}$	$(j2\pi f)^k G(f)$
Frequency Differentiation	$(-j2\pi t)^n g(t)$	$\frac{\mathrm{d}^n G(f)}{\mathrm{d}f^n}$
Complex Conjugate	$g^*(t)$	$G^*(-f)$
Time Domain Convolution	$g_1(t) * g_2(t)$	$G_1(f)G_2(f)$
Time Domain Multiplication	$g_1(t)g_2(t)$	$G_1(f) * G_2(f)$
Parseval Theorem	$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)\mathrm{d}t$	$\int_{-\infty}^{\infty} G_1(f)G_2^*(f)\mathrm{d}f$
Time Domain Integration	$\int_{-\infty}^{t} g(x)\mathrm{d}x$	$\frac{G(f)}{j2\pi f} + \frac{G(0)}{2}\delta(f)$

Formula Sheets (continued)

Table of $Q(\,)$ and $\mathrm{erf}(\,)$ functions

The approximation $Q(x) \approx \frac{1}{x\sqrt{2\pi}}(1 - \frac{0.7}{x^2})e^{-x^2/2}$ may be used when $x > 2$.

x	$\mathrm{erf}(x)$	$Q(x)$	x	$\mathrm{erf}(x)$	$Q(x)$	x	$\mathrm{erf}(x)$	$Q(x)$
0.00	0	0.5	1.70	0.9838	0.04457	3.40	1	0.0003369
0.10	0.1125	0.4602	1.80	0.9891	0.03593	3.50	1	0.0002326
0.20	0.2227	0.4207	1.90	0.9928	0.02872	3.60	1	0.0001591
0.30	0.3286	0.3821	2.00	0.9953	0.02275	3.70	1	0.0001078
0.40	0.4284	0.3446	2.10	0.997	0.01786	3.80	1	7.235×10^{-5}
0.50	0.5205	0.3085	2.20	0.9981	0.0139	3.90	1	4.810×10^{-5}
0.60	0.6039	0.2743	2.30	0.9989	0.01072	4.00	1	3.167×10^{-5}
0.70	0.6778	0.242	2.40	0.9993	0.008198	4.10	1	2.066×10^{-5}
0.80	0.7421	0.2119	2.50	0.9996	0.00621	4.20	1	1.335×10^{-5}
0.90	0.7969	0.1841	2.60	0.9998	0.004661	4.30	1	8.540×10^{-6}
1.00	0.8427	0.1587	2.70	0.9999	0.003467	4.40	1	5.413×10^{-6}
1.10	0.8802	0.1357	2.80	0.9999	0.002555	4.50	1	3.398×10^{-6}
1.20	0.9103	0.1151	2.90	1	0.001866	4.60	1	2.112×10^{-6}
1.30	0.934	0.0968	3.00	1	0.00135	4.70	1	1.301×10^{-6}
1.40	0.9523	0.08076	3.10	1	0.0009676	4.80	1	7.933×10^{-7}
1.50	0.9661	0.06681	3.20	1	0.0006871	4.90	1	4.792×10^{-7}
1.60	0.9763	0.0548	3.30	1	0.0004834	5.00	1	2.867×10^{-7}

Table of the Si( ) **function:** $\mathrm{Si}(-x) = -\mathrm{Si}(x)$, i.e. the function is odd.

x	$\mathrm{Si}(2\pi x)$	x	$\mathrm{Si}(2\pi x)$	x	$\mathrm{Si}(2\pi x)$
0.00000	0.00000	1.00000	1.41815	2.00000	1.49216
0.05000	0.31244	1.05000	1.42569	2.05000	1.49599
0.10000	0.61470	1.10000	1.44667	2.10000	1.50687
0.15000	0.89719	1.15000	1.47794	2.15000	1.52343
0.20000	1.15148	1.20000	1.51568	2.20000	1.54383
0.25000	1.37076	1.25000	1.55583	2.25000	1.56593
0.30000	1.55023	1.30000	1.59442	2.30000	1.58755
0.35000	1.68730	1.35000	1.62792	2.35000	1.60664
0.40000	1.78166	1.40000	1.65356	2.40000	1.62147
0.45000	1.83524	1.45000	1.66945	2.45000	1.63081
0.50000	1.85194	1.50000	1.67476	2.50000	1.63396
0.55000	1.83732	1.55000	1.66968	2.55000	1.63089
0.60000	1.79816	1.60000	1.65535	2.60000	1.62212
0.65000	1.74191	1.65000	1.63370	2.65000	1.60871
0.70000	1.67622	1.70000	1.60722	2.70000	1.59213
0.75000	1.60837	1.75000	1.57871	2.75000	1.57408
0.80000	1.54487	1.80000	1.55100	2.80000	1.55636
0.85000	1.49104	1.85000	1.52667	2.85000	1.54065
0.90000	1.45072	1.90000	1.50788	2.90000	1.52840
0.95000	1.42621	1.95000	1.49612	2.95000	1.52066
1.00000	1.41815	2.00000	1.49216	3.00000	1.51803