Name: ____________________ College Number: ____________________

EE501: An Introduction to the Theory of Statistical Communications

Thursday, 20 November 2014

Fourth Quiz

REMARKS:

1. Hand held calculator is allowed,
2. Open book quiz,
3. Formula sheets are attached,
4. Marks distribution:
   Question #1: 1 points
   Question #2: 2 points
   Question #3: 3 points
   Question #4: 4 points
5. Justify all your answers.

# 1	
# 2	
# 3	
# 4	

1. Let $x(t)$, $y(t)$ denote two zero-mean stationary Gaussian processes with auto-correlation functions

$$\mathscr{R}_x(\tau) = 5\mathrm{e}^{-2|\tau|}$$

$$\mathscr{R}_y(\tau) = \frac{10}{|\tau|+1}$$

(a) Calculate the average D.C. power, average A.C. power and average (total) power of each process.

(b) Is $x(t)$ ergodic? Answer could be "yes", "no", "dunno". Justify your answer.

(c) Is $y(t)$ ergodic? Answer could be "yes", "no", "dunno". Justify your answer.

Hint: The following may be useful:

$$\int \mathrm{e}^u \mathrm{d}u = \mathrm{e}^u$$

$$\int \frac{1}{u+1}\mathrm{d}u = \ln(u+1)$$

Name: ______________________ College Number: ______________________

2. Consider two zero-mean jointly Gaussian noise processes, $n_1(t)$ and $n_2(t)$ such that:

$$\begin{aligned} \mathscr{R}_i(\tau) &\triangleq \overline{n_i(t)n_i(t-\tau)} = \frac{\sin \pi\tau}{\pi\tau}; \qquad i = 1,\, 2, \\ \mathscr{R}_{12}(\tau) &\triangleq \overline{n_1(t)n_2(t-\tau)} = \frac{\sin \pi(\tau - 1/2)}{2\pi(\tau - 1/2)}\;. \end{aligned}$$

(a) Calculate the average (total) power of $n_1(t)$ and of $n_2(t)$.

(b) Calculate the mean vector and the covariance matrix of the random vector $\boldsymbol{x} = \big(n_1(0),\, n_1(1),\, n_2(0)\big)$.

Recall: $\lim_{\tau\to 0} \frac{\sin \pi\tau}{\pi\tau} = 1$.

3. Let $n_w(t)$ be a white Gaussian noise with power spectral density and autocorrelation function respectively given by:

$$\begin{aligned} S_w(f) &= \frac{\mathscr{N}_0}{2},\ \forall f \\ \mathscr{R}_w(\tau) &= \frac{\mathscr{N}_0}{2}\delta(\tau) \end{aligned}$$

Two random variables y_1, y_2 are defined as follows:

$$\begin{aligned} y_1 &= \int_{-\infty}^{\infty} h_1(t)n_w(t)\mathrm{d}t \\ y_2 &= \int_{-\infty}^{\infty} h_2(t)n_w(t)\mathrm{d}t \end{aligned}$$

where $h_1(t)$, $h_2(t)$ are two signals satisfying:

$$\begin{aligned} \int_{-\infty}^{\infty} h_1(t)^2\mathrm{d}t &< \infty \\ \int_{-\infty}^{\infty} h_2(t)^2\mathrm{d}t &< \infty \\ \int_{-\infty}^{\infty} h_1(t)h_2(t)\mathrm{d}t &= 0 \end{aligned}$$

Show that y_1, y_2 are statistically independent.
Hint: You may interchange expectations and integrals.

Name: ______________________ College Number: ______________________

4. Let $n_w(t)$ be a 0-mean stationary white Gaussian process with power spectral density $S_{n_w}(f) = 12$, $\forall f$ and autocorrelation function $\mathscr{R}_{n_w}(\tau) = 12\delta(\tau)$. We define the random process

$$n(t) = n_1 s_1(t) + n_2 s_2(t)$$

where n_1, n_2 are the random variables given by:

$$n_1 = \int_{-\infty}^{\infty} n_w(t) s_1(t) \mathrm{d}t$$
$$n_2 = \int_{-\infty}^{\infty} n_w(t) s_2(t) \mathrm{d}t$$

and

$$s_1(t) = \begin{cases} 1/2 & ; \quad 0 < t < 1 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$
$$s_2(t) = \begin{cases} t & ; \quad 0 < t < 1 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

as sketched in figure 1.

(a) Calculate the mean vector $\boldsymbol{m_n}$ and covariance matrix $\Lambda_{\boldsymbol{n}}$ of the random vector $\boldsymbol{n} = (n_1, n_2)$.
Hint: You may interchange expectations and integrals.

(b) Calculate the mean function $m_n(t)$ and the autocorrelation function $\mathscr{R}_n(t, s)$ of $n(t)$ and show that:

$$m_n(t) = 0$$

$$\mathscr{R}_n(t, s) = \begin{cases} (3 + 6(s + t) + 16st)/4 & ; \quad 0 \leq t,\ s \leq 1 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Is the process $n(t)$ wide sense stationary? Justify your answer.

(c) Calculate the probability that the random process $n(t)$ be larger than 1 at $t = 1/2$, i.e. $P(n(1/2) > 1)$.

Name: ______________________ College Number: ______________________

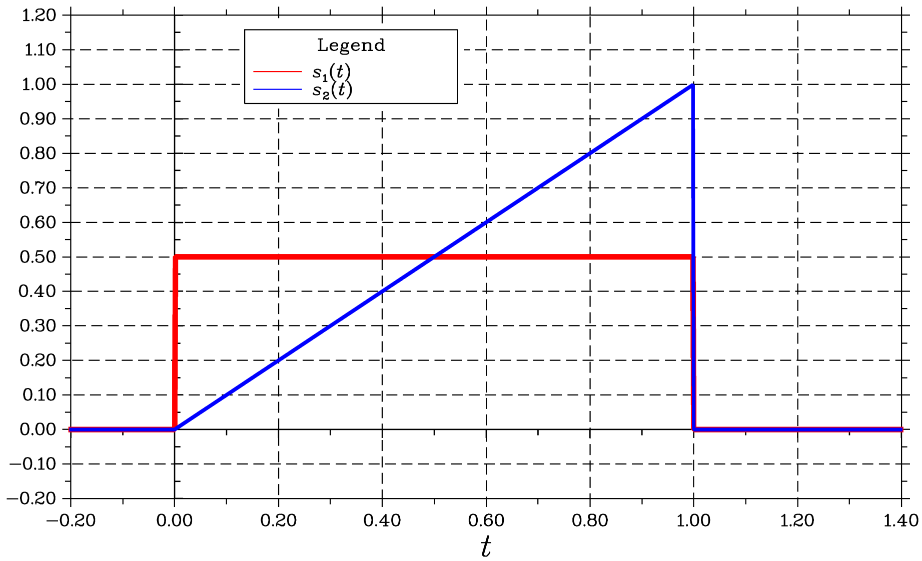

Figure 1:

END

Formula Sheets

$\cos\theta = \sin(\theta + 90°)$

$2\cos u \cos v = \cos(u-v) + \cos(u+v)$

$2\sin u \sin v = \cos(u-v) - \cos(u+v)$

$2\sin u \cos v = \sin(u-v) + \sin(u+v)$

$2\sin\theta\cos\theta = \sin(2\theta)$

$2\cos^2\theta = 1 + \cos(2\theta)$

$2\sin^2\theta = 1 - \cos(2\theta)$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

$P(AB) = P(A|B)P(B) = P(B|A)P(A)$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$p_x(\alpha) = \frac{\mathrm{d}F_x(\alpha)}{\mathrm{d}\alpha}$

$F_x(\alpha) = P(\{x \le \alpha\}) = \int_{-\infty}^{\alpha} p_x(u)\,\mathrm{d}u$

$P(a < x \le b) = \int_a^b p_x(\alpha)\mathrm{d}\alpha$

$p_x(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\alpha-\mu)^2/(2\sigma^2)}$

$$\begin{aligned} Q(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\beta^2/2}\mathrm{d}\beta \\ &= \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{\alpha}{\sqrt{2}}\right)\right] \\ &= 1 - Q(-\alpha) \end{aligned}$$

$$\begin{aligned} \mathrm{erf}(\alpha) &= \frac{2}{\sqrt{\pi}} \int_0^{\alpha} e^{-\beta^2}\mathrm{d}\beta \\ &= 1 - 2Q(\sqrt{2}\alpha) \end{aligned}$$

$P(x > a) = Q\left(\frac{a-\mu}{\sigma}\right)$

$P(a < x \le b) = Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right)$

$P(x \le a) = Q(\frac{\mu-a}{\sigma})$

$\frac{\mathrm{d}Q(\alpha)}{\mathrm{d}\alpha} = \frac{-1}{\sqrt{2\pi}} e^{-\alpha^2/2}$

$p_x(\alpha) = \int_{-\infty}^{\infty} p_{xy}(\alpha,\beta)\mathrm{d}\beta$

$y = bx + a \Rightarrow p_y(\alpha) = \frac{1}{|b|} p_x(\frac{\alpha-a}{b})$

$$p_{g(x)}(\beta) = p_y(\beta) = \begin{cases} \displaystyle\sum_{\alpha\in S(\beta)} \frac{p_x(\alpha)}{|g'(\alpha)|} & ;\ \text{if } S(\beta) \neq \emptyset \text{ and} \\ & \quad g'(\alpha) \neq 0, \forall\alpha \in S(\beta) \triangleq \{\alpha \in \mathbb{R} : \beta = g(\alpha)\} \\ 0 & ;\ \text{if } S(y) = \emptyset \end{cases}$$

$p_{\boldsymbol{f}(\boldsymbol{x})}(\boldsymbol{\beta}) = p_{\boldsymbol{y}}(\boldsymbol{\beta}) = p_{\boldsymbol{x}}(\boldsymbol{g}(\boldsymbol{\beta}))|J_{\boldsymbol{g}}(\boldsymbol{\beta})|$

$p_x(\alpha|y=v) = p_{x|y}(\alpha,v) = \frac{p_{xy}(\alpha,v)}{p_y(v)}$

x, y are independent $\Leftrightarrow p_{xy}(\alpha,\beta) = p_x(\alpha)p_y(\beta)$

Formula Sheets (continued)

Fourier Transform Properties

Operation	$g(t)$	$G(f)$
Addition	$g_1(t) + g_2(t)$	$G_1(f) + G_2(f)$
Multiplication by a constant	$ag(t)$	$aG(f)$
Symmetry	$G(t)$	$g(-f)$
Scaling	$g(at)$	$\frac{1}{\lvert a\rvert} G(\frac{f}{a})$
Time shifting	$g(t - t_0)$	$e^{-j2\pi f t_0} G(f)$
Frequency Shifting	$e^{j2\pi f_0 t} g(t)$	$G(f - f_0)$
Modulation	$2g(t)\cos(2\pi f_c t)$	$G(f - f_c) + G(f + f_c)$
Time Differentiation	$\frac{\mathrm{d}^k g(t)}{\mathrm{d}t^k}$	$(j2\pi f)^k G(f)$
Frequency Differentiation	$(-j2\pi t)^n g(t)$	$\frac{\mathrm{d}^n G(f)}{\mathrm{d}f^n}$
Complex Conjugate	$g^*(t)$	$G^*(-f)$
Time Domain Convolution	$g_1(t) * g_2(t)$	$G_1(f) G_2(f)$
Time Domain Multiplication	$g_1(t) g_2(t)$	$G_1(f) * G_2(f)$
Parseval Theorem	$\int_{-\infty}^{\infty} g_1(t) g_2^*(t) \mathrm{d}t$	$\int_{-\infty}^{\infty} G_1(f) G_2^*(f) \mathrm{d}f$
Time Domain Integration	$\int_{-\infty}^{t} g(x) \mathrm{d}x$	$\frac{G(f)}{j2\pi f} + \frac{G(0)}{2}\delta(f)$

Formula Sheets (continued)

Table of $Q(\ )$ and erf( ) functions

The approximation $Q(x) \approx \frac{1}{x\sqrt{2\pi}}(1 - \frac{0.7}{x^2})e^{-x^2/2}$ may be used when $x > 2$.

x	erf(x)	$Q(x)$	x	erf(x)	$Q(x)$	x	erf(x)	$Q(x)$
0.00	0	0.5	1.70	0.9838	0.04457	3.40	1	0.0003369
0.10	0.1125	0.4602	1.80	0.9891	0.03593	3.50	1	0.0002326
0.20	0.2227	0.4207	1.90	0.9928	0.02872	3.60	1	0.0001591
0.30	0.3286	0.3821	2.00	0.9953	0.02275	3.70	1	0.0001078
0.40	0.4284	0.3446	2.10	0.997	0.01786	3.80	1	7.235×10^{-5}
0.50	0.5205	0.3085	2.20	0.9981	0.0139	3.90	1	4.810×10^{-5}
0.60	0.6039	0.2743	2.30	0.9989	0.01072	4.00	1	3.167×10^{-5}
0.70	0.6778	0.242	2.40	0.9993	0.008198	4.10	1	2.066×10^{-5}
0.80	0.7421	0.2119	2.50	0.9996	0.00621	4.20	1	1.335×10^{-5}
0.90	0.7969	0.1841	2.60	0.9998	0.004661	4.30	1	8.540×10^{-6}
1.00	0.8427	0.1587	2.70	0.9999	0.003467	4.40	1	5.413×10^{-6}
1.10	0.8802	0.1357	2.80	0.9999	0.002555	4.50	1	3.398×10^{-6}
1.20	0.9103	0.1151	2.90	1	0.001866	4.60	1	2.112×10^{-6}
1.30	0.934	0.0968	3.00	1	0.00135	4.70	1	1.301×10^{-6}
1.40	0.9523	0.08076	3.10	1	0.0009676	4.80	1	7.933×10^{-7}
1.50	0.9661	0.06681	3.20	1	0.0006871	4.90	1	4.792×10^{-7}
1.60	0.9763	0.0548	3.30	1	0.0004834	5.00	1	2.867×10^{-7}