Name: ______________________ College Number: ______________________

**EE501: An Introduction to the Theory of Statistical Communications**

Tuesday, 22 November 2011

**Fourth Quiz**

REMARKS:
1. Hand held calculator is allowed,
2. Open book quiz, but problem solutions are not allowed,
3. A table of the $Q(\ )$ and $\mathrm{erf}(\ )$ functions is attached,
4. Marks distribution:
   Question #1: 3 points
   Question #2: 7 points
5. Justify all your answers.

| | |
|---|---|
| # 1 | |
| # 2 | |
| | |

1. Let $z(t)$ be a wide-sense stationary random process with mean $m_z$, autocorrelation function $\mathscr{R}_z(\tau)$ and power spectral density $S_z(f)$:

$$
\begin{aligned}
m_z &= 0 \\
\mathscr{R}_z(\tau) &= \frac{AW}{2}\mathrm{sinc}^2(\frac{W\tau}{2}) \\
S_z(f) &= \begin{cases} A + \frac{2Af}{W} & ; \quad -\frac{W}{2} \leq f \leq 0 \\ A - \frac{2Af}{W} & ; \quad 0 \leq f \leq \frac{W}{2} \\ 0 & ; \quad \text{elsewhere} \end{cases}
\end{aligned}
$$

The power spectral density of $z(t)$ is sketched in figure 1. As shown in figure 2, $z(t)$ is fed through a linear invariant filter (ideal low-pass filter), the frequency response of which is given by:

$$
H(f) = \begin{cases} 1 & ; \quad |f| \leq \frac{W}{4} \\ 0 & ; \quad \text{elsewhere} \end{cases}
$$

The filter's output is labelled $y(t)$.

(a) Express or sketch on a labelled graph the mean function $m_y$ and the power spectral density $S_y(f)$ of the wide-sense stationary random process $y(t)$ as functions of $A$

and $W$.

**Hint:** Do not use the expression for $\mathscr{R}_z(\tau)$ and do not use any table of Fourier transform pairs.

(b) Express the total average power of $y(t)$ as a function of $A$ and $W$.

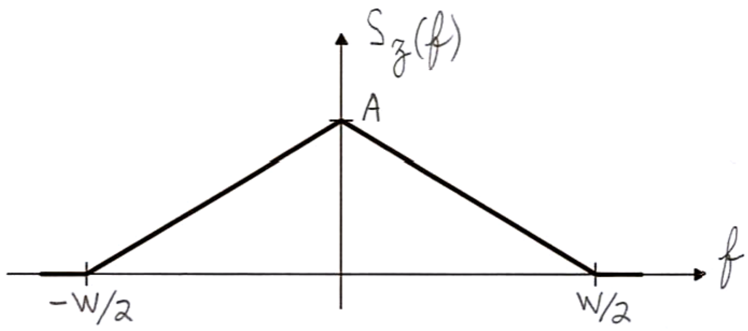


Figure 1:

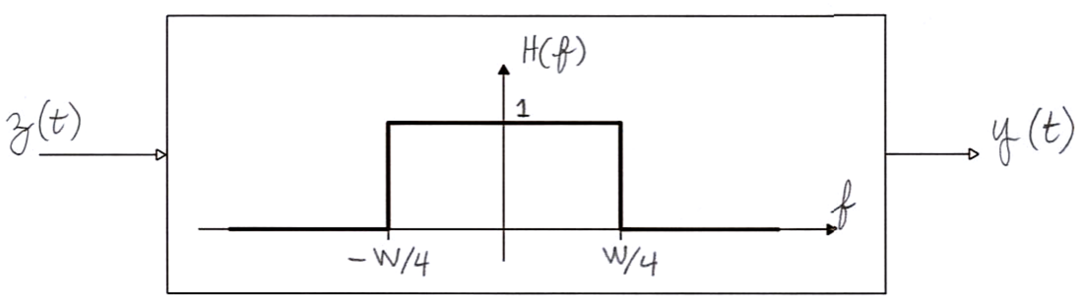


Figure 2:

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2. Let $n_w(t)$ be a 0-mean stationary white Gaussian process with power spectral density $S_{n_w}(f) = 12$, $\forall f$ and autocorrelation function $\mathscr{R}_{n_w}(\tau) = 12\delta(\tau)$. We define the random process

$$n(t) = n_1 s_1(t) + n_2 s_2(t)$$

where $n_1$, $n_2$ are the random variables given by:

$$n_1 = \int_{-\infty}^{\infty} n_w(t) s_1(t) \mathrm{d}t$$
$$n_2 = \int_{-\infty}^{\infty} n_w(t) s_2(t) \mathrm{d}t$$

and

$$s_1(t) = \begin{cases} 1/2 & ; \quad 0 < t < 1 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

$$s_2(t) = \begin{cases} t & ; \quad 0 < t < 1 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

as sketched in figure 3.

(a) Calculate the mean vector $\boldsymbol{m_n}$ and covariance matrix $\Lambda_{\boldsymbol{n}}$ of the random vector $\boldsymbol{n} = (n_1, n_2)$.
**Hint:** You may interchange expectations and integrals.

(b) Calculate the mean function $m_n(t)$ and the autocorrelation function $\mathscr{R}_n(t, s)$ of $n(t)$ and show that:

$$m_n(t) = 0$$

$$\mathscr{R}_n(t, s) = \begin{cases} (3 + 6(s + t) + 16st)/4 & ; \quad 0 \le t,\ s \le 1 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Is the process $n(t)$ wide sense stationary? Justify your answer.

(c) Calculate the probability that the random process $n(t)$ be larger than 1 at $t = 1/2$, i.e. $P(n(1/2) > 1)$.

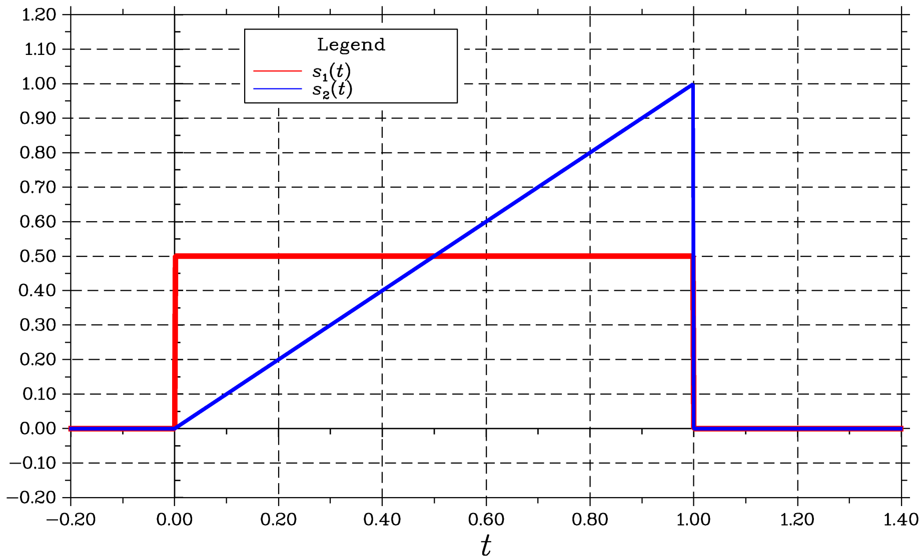
Legend
$s_1(t)$
$s_2(t)$
1.20
1.10
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
−0.10
−0.20
−0.20
0.00
0.20
0.40
0.60
0.80
1.00
1.20
1.40
$t$

Figure 3:

END

**Table of the $Q(\quad)$ and $\text{erf}(\quad)$ functions**

The approximation $Q(x) \approx \frac{1}{x\sqrt{2\pi}}(1 - \frac{0.7}{x^2})e^{-x^2/2}$ may be used when $x > 2$.

| $x$ | erf($x$) | $Q(x)$ | $x$ | erf($x$) | $Q(x)$ | $x$ | erf($x$) | $Q(x)$ |
|---|---|---|---|---|---|---|---|---|
| 0.00 | 0 | 0.5 | 1.70 | 0.9838 | 0.04457 | 3.40 | 1 | 0.0003369 |
| 0.10 | 0.1125 | 0.4602 | 1.80 | 0.9891 | 0.03593 | 3.50 | 1 | 0.0002326 |
| 0.20 | 0.2227 | 0.4207 | 1.90 | 0.9928 | 0.02872 | 3.60 | 1 | 0.0001591 |
| 0.30 | 0.3286 | 0.3821 | 2.00 | 0.9953 | 0.02275 | 3.70 | 1 | 0.0001078 |
| 0.40 | 0.4284 | 0.3446 | 2.10 | 0.997 | 0.01786 | 3.80 | 1 | $7.235 \times 10^{-5}$ |
| 0.50 | 0.5205 | 0.3085 | 2.20 | 0.9981 | 0.0139 | 3.90 | 1 | $4.810 \times 10^{-5}$ |
| 0.60 | 0.6039 | 0.2743 | 2.30 | 0.9989 | 0.01072 | 4.00 | 1 | $3.167 \times 10^{-5}$ |
| 0.70 | 0.6778 | 0.242 | 2.40 | 0.9993 | 0.008198 | 4.10 | 1 | $2.066 \times 10^{-5}$ |
| 0.80 | 0.7421 | 0.2119 | 2.50 | 0.9996 | 0.00621 | 4.20 | 1 | $1.335 \times 10^{-5}$ |
| 0.90 | 0.7969 | 0.1841 | 2.60 | 0.9998 | 0.004661 | 4.30 | 1 | $8.540 \times 10^{-6}$ |
| 1.00 | 0.8427 | 0.1587 | 2.70 | 0.9999 | 0.003467 | 4.40 | 1 | $5.413 \times 10^{-6}$ |
| 1.10 | 0.8802 | 0.1357 | 2.80 | 0.9999 | 0.002555 | 4.50 | 1 | $3.398 \times 10^{-6}$ |
| 1.20 | 0.9103 | 0.1151 | 2.90 | 1 | 0.001866 | 4.60 | 1 | $2.112 \times 10^{-6}$ |
| 1.30 | 0.934 | 0.0968 | 3.00 | 1 | 0.00135 | 4.70 | 1 | $1.301 \times 10^{-6}$ |
| 1.40 | 0.9523 | 0.08076 | 3.10 | 1 | 0.0009676 | 4.80 | 1 | $7.933 \times 10^{-7}$ |
| 1.50 | 0.9661 | 0.06681 | 3.20 | 1 | 0.0006871 | 4.90 | 1 | $4.792 \times 10^{-7}$ |
| 1.60 | 0.9763 | 0.0548 | 3.30 | 1 | 0.0004834 | 5.00 | 1 | $2.867 \times 10^{-7}$ |