

Name: _____

College Number: _____

EE501: An Introduction to the Theory of Statistical Communications

Tuesday, 3 November 2015

Third Quiz

- REMARKS:
1. Hand held calculator is allowed,
 2. Open book quiz,
 3. Formula sheets are attached,
 4. Marks distribution:
 - Question #1: 2 points
 - Question #2: 3 points
 - Question #3: 3 points
 - Question #4: 2 points
 5. Justify all your answers.

# 1	
# 2	
# 3	
# 4	

1. Let y be a Rayleigh random variable (2nd type of random variables described in §2.10.2) and x is another random variable that need not be statistically independent of y . The densities $p_x(\alpha)$, $p_{xy}(\alpha, \beta)$ are not known. Calculate the missing values in the following:

$$E[x] = 2$$

$$E[y] = ?$$

$$E[x^2] = 5$$

$$E[y^2] = ?$$

$$Var(x) = ?$$

$$Var(y) = \pi \left(1 - \frac{\pi}{4}\right)$$

$$E[xy] = \sqrt{\pi} (\sqrt{\pi} - 1/2)$$

$$Cov(x, y) = ?$$

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2. Let $x_1, x_2, \dots, x_{1000}$ denote 1000 statistically independent identically distributed random variables. x_i is a uniform random variable between -1.5 and +1 for every $i = 1, 2, \dots, 1000$. The sample mean m of $x_1, x_2, \dots, x_{1000}$ is defined by:

$$m = \frac{1}{1000} \sum_{i=1}^{1000} x_i$$

- (a) Using the Weak Law of Large Numbers, estimate the range within which m lies with a probability of at least 95%. In other words, use the Weak Law of Large Numbers to find a, b such that $P(a < m < b) \geq 0.95$.

Remark: You can also do this with Chebyshev's inequality and find the same answer.

- (b) Using the expression:

$$P(|m - \bar{x}| \geq \epsilon) \approx 2Q\left(\frac{\epsilon\sqrt{N}}{\sigma_x}\right)$$

obtained from the Central Limit Theorem, estimate the range within which m lies with a probability of approximately 95%.

3. Let $\mathbf{x} = [x_1 \ x_2 \ x_3]$ be a non-singular Gaussian random vector with mean vector and covariance matrix respectively given by:

$$\mathbf{m}_x = [0 \ 1/2 \ 1]$$
$$\Lambda_x = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

- (a) Calculate the probability density function of the random variable x_3 and calculate the probability of the event $\{x_3 > 2.5\}$.
- (b) Calculate the probability of the event $\{2x_1 - 3x_3 > 10\}$.

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4. Consider the stationary random process¹ $x(t) = 10 \sin(2\pi t + \theta)$ where θ is a random variable uniformly distributed between 0 and 2π . What is the probability of the set of sample functions passing through the window in figure 1?

Hints:

- Use the probability density function of example 3.1.2. Justify all the steps of your solution.
- The following may be useful:

$$\begin{aligned} \sin(u + \pi/2) &= \cos(u) \\ \int \sin^3(t) dt &= -\frac{\sin^2(t) \cos(t) + 2 \cos(t)}{3} \\ \int \cos(t) \sin^2(t) dt &= \sin^3(t)/3 \\ \int \cos^2(t) \sin(t) dt &= -\cos^3(t)/3 \\ \int \cos^3(t) dt &= \frac{\sin(t) \cos^2(t) + 2 \sin(t)}{3} \\ \int \frac{1}{\pi \sqrt{100 - u^2}} du &= \frac{1}{\pi} \arcsin\left(\frac{u}{10}\right) \end{aligned}$$

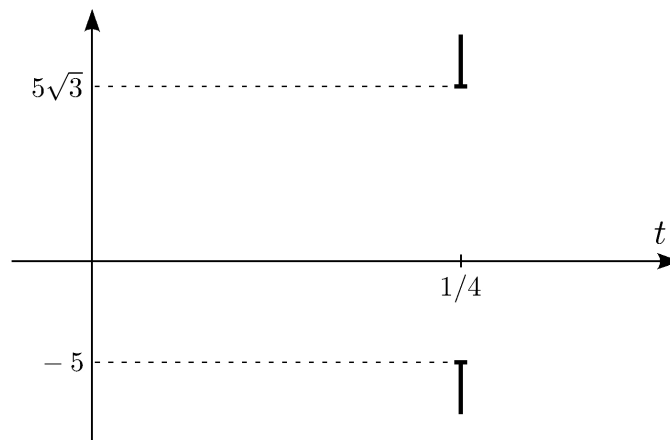


Figure 1:

END

¹This is the process described in example 3.1.2 of the notes.

Formula Sheets

$$\cos \theta = \sin(\theta + 90^\circ)$$

$$2 \sin u \sin v = \cos(u - v) - \cos(u + v)$$

$$2 \sin \theta \cos \theta = \sin(2\theta)$$

$$2 \sin^2 \theta = 1 - \cos(2\theta)$$

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$F_x(\alpha) = P(\{x \leq \alpha\}) = \int_{-\infty}^{\alpha} p_x(u) du$$

$$p_x(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\alpha-\mu)^2/(2\sigma^2)}$$

$$\begin{aligned} \operatorname{erf}(\alpha) &= \frac{2}{\sqrt{\pi}} \int_0^{\alpha} e^{-\beta^2} d\beta \\ &= 1 - 2Q(\sqrt{2}\alpha) \end{aligned}$$

$$P(a < x \leq b) = Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right)$$

$$\frac{dQ(\alpha)}{d\alpha} = \frac{-1}{\sqrt{2\pi}} e^{-\alpha^2/2}$$

$$y = bx + a \Rightarrow p_y(\alpha) = \frac{1}{|b|} p_x\left(\frac{\alpha-a}{b}\right)$$

$$p_{g(x)}(\beta) = p_y(\beta) = \begin{cases} \sum_{\alpha \in S(\beta)} \frac{p_x(\alpha)}{|g'(\alpha)|} & ; \text{ if } S(\beta) \neq \emptyset \text{ and} \\ & g'(\alpha) \neq 0, \forall \alpha \in S(\beta) \triangleq \{\alpha \in \mathbb{R} : \beta = g(\alpha)\} \\ 0 & ; \text{ if } S(y) = \emptyset \end{cases}$$

$$p_{\mathbf{f}(\mathbf{x})}(\boldsymbol{\beta}) = p_{\mathbf{y}}(\boldsymbol{\beta}) = p_{\mathbf{x}}(\mathbf{g}(\boldsymbol{\beta})) |J_{\mathbf{g}}(\boldsymbol{\beta})|$$

$$x, y \text{ are independent} \Leftrightarrow p_{xy}(\alpha, \beta) = p_x(\alpha)p_y(\beta)$$

$$2 \cos u \cos v = \cos(u - v) + \cos(u + v)$$

$$2 \sin u \cos v = \sin(u - v) + \sin(u + v)$$

$$2 \cos^2 \theta = 1 + \cos(2\theta)$$

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$P(AB) = P(A|B)P(B) = P(B|A)P(A)$$

$$p_x(\alpha) = \frac{dF_x(\alpha)}{d\alpha}$$

$$P(a < x \leq b) = \int_a^b p_x(\alpha) d\alpha$$

$$\begin{aligned} Q(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\beta^2/2} d\beta \\ &= \frac{1}{2} \left[1 - \operatorname{erf}\left(\frac{\alpha}{\sqrt{2}}\right) \right] \\ &= 1 - Q(-\alpha) \end{aligned}$$

$$P(x > a) = Q\left(\frac{a-\mu}{\sigma}\right)$$

$$P(x \leq a) = Q\left(\frac{\mu-a}{\sigma}\right)$$

$$p_x(\alpha) = \int_{-\infty}^{\infty} p_{xy}(\alpha, \beta) d\beta$$

$$p_x(\alpha|y = v) = p_{x|y}(\alpha, v) = \frac{p_{xy}(\alpha, v)}{p_y(v)}$$

Formula Sheets (continued)

Fourier Transform Properties

Operation	$g(t)$	$G(f)$
Addition	$g_1(t) + g_2(t)$	$G_1(f) + G_2(f)$
Multiplication by a constant	$ag(t)$	$aG(f)$
Symmetry	$G(t)$	$g(-f)$
Scaling	$g(at)$	$\frac{1}{ a }G\left(\frac{f}{a}\right)$
Time shifting	$g(t - t_0)$	$e^{-j2\pi ft_0}G(f)$
Frequency Shifting	$e^{j2\pi f_0 t}g(t)$	$G(f - f_0)$
Modulation	$2g(t) \cos(2\pi f_c t)$	$G(f - f_c) + G(f + f_c)$
Time Differentiation	$\frac{d^k g(t)}{dt^k}$	$(j2\pi f)^k G(f)$
Frequency Differentiation	$(-j2\pi t)^n g(t)$	$\frac{d^n G(f)}{df^n}$
Complex Conjugate	$g^*(t)$	$G^*(-f)$
Time Domain Convolution	$g_1(t) * g_2(t)$	$G_1(f)G_2(f)$
Time Domain Multiplication	$g_1(t)g_2(t)$	$G_1(f) * G_2(f)$
Parseval Theorem	$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)dt$	$\int_{-\infty}^{\infty} G_1(f)G_2^*(f)df$
Time Domain Integration	$\int_{-\infty}^t g(x)dx$	$\frac{G(f)}{j2\pi f} + \frac{G(0)}{2}\delta(f)$

Formula Sheets (continued)

Table of $Q(\cdot)$ and $\text{erf}(\cdot)$ functions

The approximation $Q(x) \approx \frac{1}{x\sqrt{2\pi}}(1 - \frac{0.7}{x^2})e^{-x^2/2}$ may be used when $x > 2$.

x	$\text{erf}(x)$	$Q(x)$	x	$\text{erf}(x)$	$Q(x)$	x	$\text{erf}(x)$	$Q(x)$
0.00	0	0.5	1.70	0.9838	0.04457	3.40	1	0.0003369
0.10	0.1125	0.4602	1.80	0.9891	0.03593	3.50	1	0.0002326
0.20	0.2227	0.4207	1.90	0.9928	0.02872	3.60	1	0.0001591
0.30	0.3286	0.3821	2.00	0.9953	0.02275	3.70	1	0.0001078
0.40	0.4284	0.3446	2.10	0.997	0.01786	3.80	1	7.235×10^{-5}
0.50	0.5205	0.3085	2.20	0.9981	0.0139	3.90	1	4.810×10^{-5}
0.60	0.6039	0.2743	2.30	0.9989	0.01072	4.00	1	3.167×10^{-5}
0.70	0.6778	0.242	2.40	0.9993	0.008198	4.10	1	2.066×10^{-5}
0.80	0.7421	0.2119	2.50	0.9996	0.00621	4.20	1	1.335×10^{-5}
0.90	0.7969	0.1841	2.60	0.9998	0.004661	4.30	1	8.540×10^{-6}
1.00	0.8427	0.1587	2.70	0.9999	0.003467	4.40	1	5.413×10^{-6}
1.10	0.8802	0.1357	2.80	0.9999	0.002555	4.50	1	3.398×10^{-6}
1.20	0.9103	0.1151	2.90	1	0.001866	4.60	1	2.112×10^{-6}
1.30	0.934	0.0968	3.00	1	0.00135	4.70	1	1.301×10^{-6}
1.40	0.9523	0.08076	3.10	1	0.0009676	4.80	1	7.933×10^{-7}
1.50	0.9661	0.06681	3.20	1	0.0006871	4.90	1	4.792×10^{-7}
1.60	0.9763	0.0548	3.30	1	0.0004834	5.00	1	2.867×10^{-7}