Name: ____________________ College Number: ____________________

EE501: An Introduction to the Theory of Statistical Communications

Thursday, 23 October 2014

Third Quiz

REMARKS:
1. Hand held calculator is allowed,
2. Open book quiz,
3. Formula sheets are attached,
4. Marks distribution:
   Question #1: 5 points
   Question #2: 4 points
   Question #3: 1 points
5. Justify all your answers.

# 1	
# 2	
# 3	

1. Let x_1, x_2, ..., x_{1000} denote 1000 statistically independent identically distributed random variables with mean and variance respectively denoted by $\bar{x}$, σ_x^2. x_i takes the values -1, 0, 1 with probabilities:

$$\begin{aligned} P(x_i = -1) &= 1/2, \\ P(x_i = 0) &= 1/4, \\ P(x_i = 1) &= 1/4, \end{aligned}$$

for every $i = 1, 2, \ldots, 1000$. The sample mean m of x_1, x_2, ..., x_{1000} is defined by:

$$m = \frac{1}{1000} \sum_{i=1}^{1000} x_i$$

(a) Using the Weak Law of Large Numbers, estimate the range within which m lies with a probability of at least 95%. In other words, use the Weak Law of Large Numbers to find a, b such that $P(a < m < b) \geq 0.95$.
Remark: You can also do this with Chebyshev's inequality and find the same answer.

(b) Using the expression:

$$P\left(\left|m - \bar{x}\right| \geq \epsilon\right) \approx 2Q\left(\frac{\epsilon\sqrt{N}}{\sigma_x}\right)$$

obtained from the Central Limit Theorem, estimate the range within which m lies with a probability of approximately 95%.

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2. The mean vector and covariance matrix of a random vector $\boldsymbol{x} = (x_1,\, x_2)$ are respectively given by:

$$\boldsymbol{m_x} = (1,\, 0)$$

$$\Lambda_{\boldsymbol{x}} = \begin{pmatrix} 9 & -4 \\ -4 & 4 \end{pmatrix}$$

(a) Calculate the mean vector $\boldsymbol{m_y}$ and covariance matrix $\Lambda_{\boldsymbol{y}}$ of the random vector $\boldsymbol{y} = (y_1,\, y_2)$ defined by the transformation:

$$\boldsymbol{y} = \boldsymbol{x} \times \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} + (0,\, 1)$$

(b) If the above random vector $\boldsymbol{x}$ is Gaussian then $\boldsymbol{y}$ is also Gaussian. Calculate the probability that $\boldsymbol{y} = (y_1,\, y_2)$ lies in the hatched region of the plane in figure 1.

Note: If you have not been able to solve question (2a), you may use the following:

$$\boldsymbol{m_y} = (2,\, 1)$$

$$\Lambda_{\boldsymbol{y}} = \begin{pmatrix} 25 & -10 \\ -10 & 15 \end{pmatrix}$$

Those *are not* the answer to question (2a).

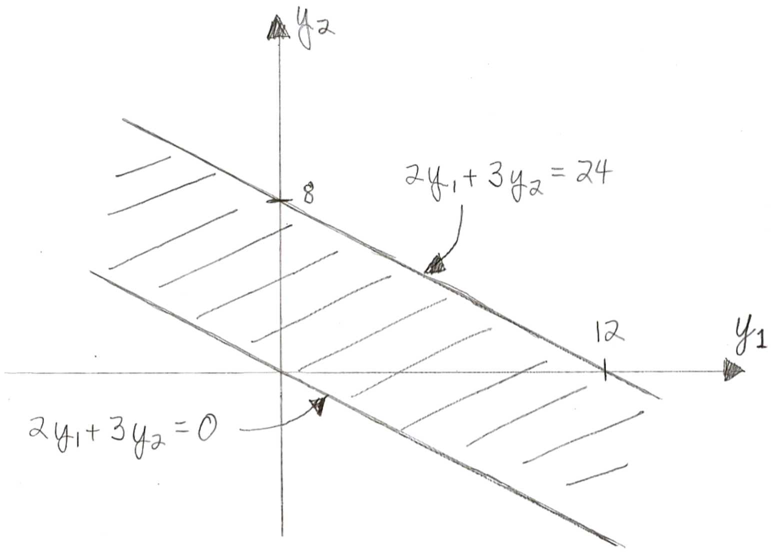

Figure 1:

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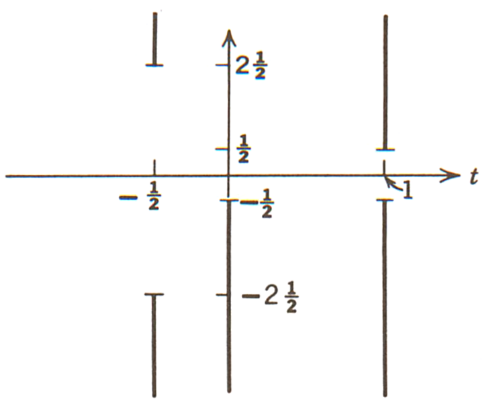
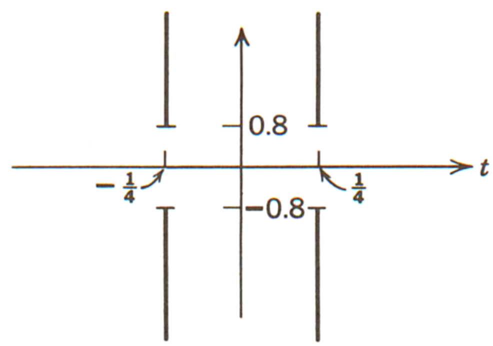
3. An elementary random process comprises four sample functions, to each of which is assigned equal probability.

$$x(\omega_1,\, t) = 1 \qquad\qquad x(\omega_3,\, t) = \sin\ \pi t\,,$$
$$x(\omega_2,\, t) = -2 \qquad\qquad x(\omega_4,\, t) = \cos\ \pi t\,.$$

What is the probability of the set of sample functions passing through the windows of figure 2(a)? — Fig. 2(b)?



(a)



(b)

Figure 2:

END

Formula Sheets

$\cos\theta = \sin(\theta + 90^\circ)$

$2\cos u \cos v = \cos(u-v) + \cos(u+v)$

$2\sin u \sin v = \cos(u-v) - \cos(u+v)$

$2\sin u \cos v = \sin(u-v) + \sin(u+v)$

$2\sin\theta\cos\theta = \sin(2\theta)$

$2\cos^2\theta = 1 + \cos(2\theta)$

$2\sin^2\theta = 1 - \cos(2\theta)$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

$P(AB) = P(A|B)P(B) = P(B|A)P(A)$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$p_x(\alpha) = \frac{\mathrm{d}F_x(\alpha)}{\mathrm{d}\alpha}$

$F_x(\alpha) = P(\{x \le \alpha\}) = \int_{-\infty}^{\alpha} p_x(u)\,\mathrm{d}u$

$P(a < x \le b) = \int_a^b p_x(\alpha)\mathrm{d}\alpha$

$p_x(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\alpha-\mu)^2/(2\sigma^2)}$

$$\begin{aligned} Q(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\beta^2/2}\mathrm{d}\beta \\ &= \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{\alpha}{\sqrt{2}}\right)\right] \\ &= 1 - Q(-\alpha) \end{aligned}$$

$$\begin{aligned} \mathrm{erf}(\alpha) &= \frac{2}{\sqrt{\pi}} \int_0^{\alpha} e^{-\beta^2}\mathrm{d}\beta \\ &= 1 - 2Q(\sqrt{2}\alpha) \end{aligned}$$

$P(x > a) = Q\left(\frac{a-\mu}{\sigma}\right)$

$P(a < x \le b) = Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right)$

$P(x \le a) = Q(\frac{\mu-a}{\sigma})$

$\frac{\mathrm{d}Q(\alpha)}{\mathrm{d}\alpha} = \frac{-1}{\sqrt{2\pi}} e^{-\alpha^2/2}$

$p_x(\alpha) = \int_{-\infty}^{\infty} p_{xy}(\alpha,\beta)\mathrm{d}\beta$

$y = bx + a \Rightarrow p_y(\alpha) = \frac{1}{|b|} p_x(\frac{\alpha-a}{b})$

$$p_{g(x)}(\beta) = p_y(\beta) = \begin{cases} \sum\limits_{\alpha\in S(\beta)} \frac{p_x(\alpha)}{|g'(\alpha)|} & ; \text{ if } S(\beta) \ne \emptyset \text{ and } \\ & \quad g'(\alpha) \ne 0, \forall\alpha \in S(\beta) \triangleq \{\alpha \in \mathbb{R} : \beta = g(\alpha)\} \\ 0 & ; \text{ if } S(y) = \emptyset \end{cases}$$

$p_{\boldsymbol{f}(\boldsymbol{x})}(\boldsymbol{\beta}) = p_{\boldsymbol{y}}(\boldsymbol{\beta}) = p_{\boldsymbol{x}}(\boldsymbol{g}(\boldsymbol{\beta}))|J_{\boldsymbol{g}}(\boldsymbol{\beta})|$

$p_x(\alpha|y=v) = p_{x|y}(\alpha,v) = \frac{p_{xy}(\alpha,v)}{p_y(v)}$

x, y are independent $\Leftrightarrow p_{xy}(\alpha,\beta) = p_x(\alpha)p_y(\beta)$

Formula Sheets (continued)

Fourier Transform Properties

Operation	$g(t)$	$G(f)$
Addition	$g_1(t) + g_2(t)$	$G_1(f) + G_2(f)$
Multiplication by a constant	$ag(t)$	$aG(f)$
Symmetry	$G(t)$	$g(-f)$
Scaling	$g(at)$	$\frac{1}{\|a\|}G(\frac{f}{a})$
Time shifting	$g(t - t_0)$	$e^{-j2\pi f t_0}G(f)$
Frequency Shifting	$e^{j2\pi f_0 t}g(t)$	$G(f - f_0)$
Modulation	$2g(t)\cos(2\pi f_c t)$	$G(f - f_c) + G(f + f_c)$
Time Differentiation	$\frac{\mathrm{d}^k g(t)}{\mathrm{d}t^k}$	$(j2\pi f)^k G(f)$
Frequency Differentiation	$(-j2\pi t)^n g(t)$	$\frac{\mathrm{d}^n G(f)}{\mathrm{d}f^n}$
Complex Conjugate	$g^*(t)$	$G^*(-f)$
Time Domain Convolution	$g_1(t) * g_2(t)$	$G_1(f)G_2(f)$
Time Domain Multiplication	$g_1(t)g_2(t)$	$G_1(f) * G_2(f)$
Parseval Theorem	$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)\mathrm{d}t$	$\int_{-\infty}^{\infty} G_1(f)G_2^*(f)\mathrm{d}f$
Time Domain Integration	$\int_{-\infty}^{t} g(x)\mathrm{d}x$	$\frac{G(f)}{j2\pi f} + \frac{G(0)}{2}\delta(f)$

Formula Sheets (continued)

Table of $Q(\,)$ and $\text{erf}(\,)$ functions

The approximation $Q(x) \approx \frac{1}{x\sqrt{2\pi}}(1 - \frac{0.7}{x^2})e^{-x^2/2}$ may be used when $x > 2$.

x	$\text{erf}(x)$	$Q(x)$	x	$\text{erf}(x)$	$Q(x)$	x	$\text{erf}(x)$	$Q(x)$
0.00	0	0.5	1.70	0.9838	0.04457	3.40	1	0.0003369
0.10	0.1125	0.4602	1.80	0.9891	0.03593	3.50	1	0.0002326
0.20	0.2227	0.4207	1.90	0.9928	0.02872	3.60	1	0.0001591
0.30	0.3286	0.3821	2.00	0.9953	0.02275	3.70	1	0.0001078
0.40	0.4284	0.3446	2.10	0.997	0.01786	3.80	1	7.235×10^{-5}
0.50	0.5205	0.3085	2.20	0.9981	0.0139	3.90	1	4.810×10^{-5}
0.60	0.6039	0.2743	2.30	0.9989	0.01072	4.00	1	3.167×10^{-5}
0.70	0.6778	0.242	2.40	0.9993	0.008198	4.10	1	2.066×10^{-5}
0.80	0.7421	0.2119	2.50	0.9996	0.00621	4.20	1	1.335×10^{-5}
0.90	0.7969	0.1841	2.60	0.9998	0.004661	4.30	1	8.540×10^{-6}
1.00	0.8427	0.1587	2.70	0.9999	0.003467	4.40	1	5.413×10^{-6}
1.10	0.8802	0.1357	2.80	0.9999	0.002555	4.50	1	3.398×10^{-6}
1.20	0.9103	0.1151	2.90	1	0.001866	4.60	1	2.112×10^{-6}
1.30	0.934	0.0968	3.00	1	0.00135	4.70	1	1.301×10^{-6}
1.40	0.9523	0.08076	3.10	1	0.0009676	4.80	1	7.933×10^{-7}
1.50	0.9661	0.06681	3.20	1	0.0006871	4.90	1	4.792×10^{-7}
1.60	0.9763	0.0548	3.30	1	0.0004834	5.00	1	2.867×10^{-7}