Name: ____________________ College Number: ____________________

EE501: An Introduction to the Theory of Statistical Communications

Tuesday, 1 November 2011

Third Quiz

REMARKS:

1. Hand held calculator is allowed,
2. Open book quiz, but problem solutions are not allowed,
3. A table of the $Q(\ )$ and $\text{erf}(\ )$ functions is attached,
4. Marks distribution:
   Question #1: 2 points
   Question #2: 3 points
   Question #3: 5 points
5. Justify all your answers.

# 1	
# 2	
# 3	

1. Consider a random variable x with mean $\bar{x} = 2$ and second moment $\overline{x^2} = 24$. Using Chebyshev's inequality, estimate the range within which x lies with a probability of at least 95%. In other words, use Chebyshev's inequality to find a, b such that $P(a < x < b) \geq 0.95$.

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2. The mean vector and covariance matrix of a random vector $\boldsymbol{x} = (x_1,\, x_2)$ are respectively given by:

$$\boldsymbol{m_x} = (1,\, 0)$$

$$\Lambda_{\boldsymbol{x}} = \begin{pmatrix} 9 & -4 \\ -4 & 4 \end{pmatrix}$$

(a) Calculate the mean vector $\boldsymbol{m_y}$ and covariance matrix $\Lambda_{\boldsymbol{y}}$ of the random vector $\boldsymbol{y} = (y_1,\, y_2)$ defined by the transformation:

$$\boldsymbol{y} = \boldsymbol{x} \times \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} + (0,\, 1)$$

(b) If the above random vector $\boldsymbol{x}$ is Gaussian then $\boldsymbol{y}$ is also Gaussian. Calculate the probability that $\boldsymbol{y} = (y_1,\, y_2)$ lies in the hatched region of the plane in figure 1.

Note: If you have not been able to solve question (2a), you may use the following:

$$\boldsymbol{m_y} = (2,\, 1)$$

$$\Lambda_{\boldsymbol{y}} = \begin{pmatrix} 25 & -10 \\ -10 & 15 \end{pmatrix}$$

Those *are not* the answer to question (2a).

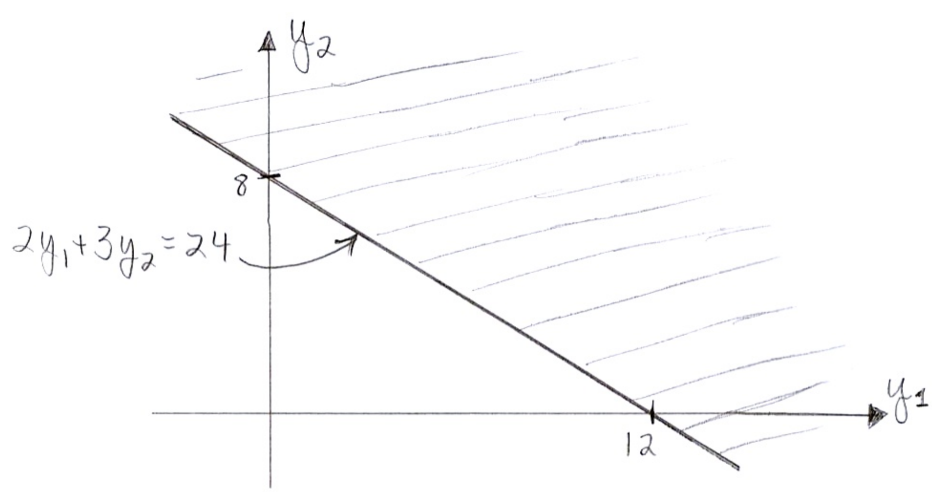

Figure 1:

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3. A communication system is used to transmit a voltage s to a decision device as shown in figure 2. The channel output is a continuous random variable $r = s + n$ where n is a Cauchy distributed random variable[1] statistically independent of s and with probability density function

$$p_n(\alpha) = \frac{b/\pi}{\alpha^2 + b^2}\,.$$

The system is used to communicate one of two equally likely messages m_0 and m_1 with corresponding signals

$$\begin{aligned} s_0 &= 3b/2 \\ s_1 &= -b/2 \end{aligned}$$

(a) Determine the optimum receiver decision rule.

(b) Compute the resulting probability of error with the decision rule found in 3a.

Recall:

$$\begin{aligned} a^2 - b^2 &= (a-b)(a+b) \\ \int \frac{b/\pi}{u^2+b^2}\mathrm{d}u &= \frac{1}{\pi}\arctan\left(\frac{u}{b}\right) + \text{constant, for any } b > 0 \end{aligned}$$

$$\begin{aligned} \arctan(\infty) &= \pi/2 \\ \arctan(-\infty) &= -\pi/2 \end{aligned} \qquad \begin{aligned} \arctan(0) &= 0 \\ \arctan(1) &= \pi/4 \\ \arctan(-1) &= -\pi/4 \end{aligned}$$

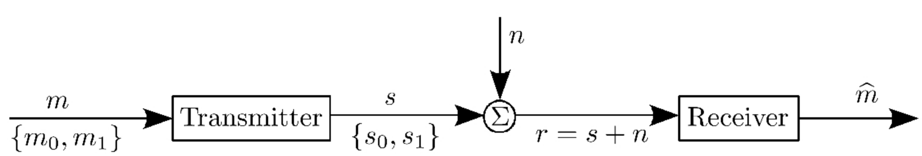

Figure 2:

END

[1] Refer to Wozencraft & Jacobs, page 48, equations (2.48a), (2.48b).

Table of the $Q(\quad)$ and $\mathrm{erf}(\quad)$ functions

The approximation $Q(x) \approx \frac{1}{x\sqrt{2\pi}}(1 - \frac{0.7}{x^2})e^{-x^2/2}$ may be used when $x > 2$.

x	erf(x)	$Q(x)$	x	erf(x)	$Q(x)$	x	erf(x)	$Q(x)$
0.00	0	0.5	1.70	0.9838	0.04457	3.40	1	0.0003369
0.10	0.1125	0.4602	1.80	0.9891	0.03593	3.50	1	0.0002326
0.20	0.2227	0.4207	1.90	0.9928	0.02872	3.60	1	0.0001591
0.30	0.3286	0.3821	2.00	0.9953	0.02275	3.70	1	0.0001078
0.40	0.4284	0.3446	2.10	0.997	0.01786	3.80	1	7.235×10^{-5}
0.50	0.5205	0.3085	2.20	0.9981	0.0139	3.90	1	4.810×10^{-5}
0.60	0.6039	0.2743	2.30	0.9989	0.01072	4.00	1	3.167×10^{-5}
0.70	0.6778	0.242	2.40	0.9993	0.008198	4.10	1	2.066×10^{-5}
0.80	0.7421	0.2119	2.50	0.9996	0.00621	4.20	1	1.335×10^{-5}
0.90	0.7969	0.1841	2.60	0.9998	0.004661	4.30	1	8.540×10^{-6}
1.00	0.8427	0.1587	2.70	0.9999	0.003467	4.40	1	5.413×10^{-6}
1.10	0.8802	0.1357	2.80	0.9999	0.002555	4.50	1	3.398×10^{-6}
1.20	0.9103	0.1151	2.90	1	0.001866	4.60	1	2.112×10^{-6}
1.30	0.934	0.0968	3.00	1	0.00135	4.70	1	1.301×10^{-6}
1.40	0.9523	0.08076	3.10	1	0.0009676	4.80	1	7.933×10^{-7}
1.50	0.9661	0.06681	3.20	1	0.0006871	4.90	1	4.792×10^{-7}
1.60	0.9763	0.0548	3.30	1	0.0004834	5.00	1	2.867×10^{-7}