Name: ____________________ College Number: ____________________

EE501: An Introduction to the Theory of Statistical Communications

Thursday, 28 October 2010

Third Quiz

REMARKS:
1. Hand held calculator is allowed,
2. Open book quiz, but problem solutions are not allowed,
3. A table of the $Q(\ )$ and erf( ) functions is attached,
4. Marks distribution:
   Question #1: 5 points
   Question #2: 6 points
   Question #3: 4 points
   Question #4: 5 points
5. Justify all your answers.

# 1	
# 2	
# 3	
# 4	

1. [1] A communication system is used to transmit one of two equally likely messages, m_0 and m_1. The channel output is a continuous random variable r, the conditional density functions of which are shown in Fig. 1. Determine the optimum receiver decision rule and compute the resulting probability of error.

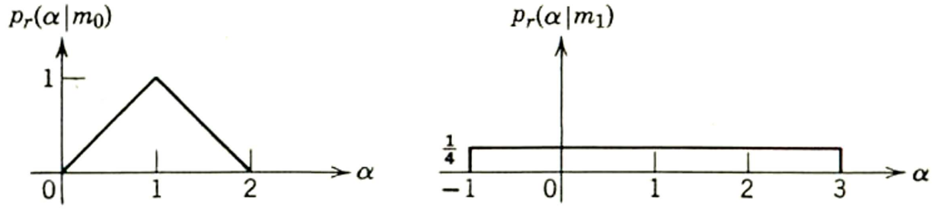

Figure 1:

[1]Problem 2.23 in Wozencraft & Jacobs, page 121.

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2. [2] We wish to simulate a communication system on a digital computer and estimate the error probability $P(\mathscr{E})$ by measuring the relative frequency of error. Let N denote the number of independent uses of the channel in the simulation and x_i, $i = 1, 2, \ldots, N$ denote the random variable such that:

$$x_i = \begin{cases} 0 & ; \text{ no error on the } i\text{-th use of the channel} \\ \\ 1 & ; \text{ there is an error on the } i\text{-th use of the channel} \end{cases}$$

The x_i are identically distributed and statistically independent random variables. The relative frequency of error, i.e. the estimate of $P(\mathscr{E})$ is given by the sample mean:

$$m = \frac{1}{N}\sum_{i=1}^{N} x_i$$

and clearly $E[m] = E[x_i] = P(\mathscr{E})$. Approximate by means of

(a) the Weak Law of Large Numbers,

(b) the Central Limit Theorem,

how many independent uses of the channel we must simulate in order to be 99.9% certain that the observed relative frequency lies within 5% of the true $P[\mathscr{E}]$, which we may assume to be approximately 0.01.

Hint: For the Weak Law of Large Numbers, use equation (2.19) on page 62 in the notes (in theorem 27). For the Central Limit Theorem, use the approximation:

$$P\left(\left|\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right) - \bar{x}\right| \geq \epsilon\right) \approx 2Q\left(\frac{\epsilon\sqrt{N}}{\sigma_x}\right)$$

which follows from equation (2.20) on page 63 in the notes.

3. Let x, y be two statistically independent Gaussian random variables with means and variances respectively given by:

$$\begin{array}{rclcrcl} \overline{x} & = & 2 & \quad & Var(x) & = & 7 \\ \overline{y} & = & -3 & \quad & Var(y) & = & 9 \end{array}$$

The random variable z is defined by:

$$z = x + y$$

Calculate $P(z > 5)$, i.e. the probability of the event $\{x + y > 5\}$.

Hint: Theorem 25, proposition 22 and the equations on page 21 of the course notes.

[2] This problem is based on problem 2.38, page 127 in W&J.

4. The probability density function and characteristic function of a random variable y are respectively given by:

$$\begin{aligned} p_y(\beta) &= \mathrm{e}^{-2|\beta|} \\ M_y(\nu) &= \frac{4}{4+\nu^2} \end{aligned}$$

By symmetry of $p_y(\beta)$ one easily sees that $\overline{y} = 0$.

(a) Calculate the variance, σ_y^2, of the random variable y.

(b) For the values of $N = 1, 2, 5, 10$ sketch the function $M_y(\nu/\sqrt{N})^N$ on the same graph for $-15 \leq \nu \leq 15$.

(c) From the central limit theorem, we know that as N increases, the function $M_y(\nu/\sqrt{N})^N$ becomes *closer and closer* to the function $\mathrm{e}^{-\zeta\nu^2}$ for some value $\zeta \in \mathbb{R}$. What is the value of ζ? Sketch $\mathrm{e}^{-\zeta\nu^2}$ for $-15 \leq \nu \leq 15$ on the graph of part 4b above (to confirm your answer).

Recall: The following may (or may not) be useful:

$$\begin{aligned} \int \mathrm{e}^{-2u}\mathrm{d}u &= -\frac{1}{2}\,\mathrm{e}^{-2u} + \text{ constant} \\ \int u\mathrm{e}^{-2u}\mathrm{d}u &= -\frac{1}{4}(1+2u)\,\mathrm{e}^{-2u} + \text{ constant} \\ \int u^2\mathrm{e}^{-2u}\mathrm{d}u &= -\frac{1}{4}(1+2u+2u^2)\,\mathrm{e}^{-2u} + \text{ constant} \end{aligned}$$

$$\begin{aligned} \int \mathrm{e}^{2u}\mathrm{d}u &= \frac{1}{2}\,\mathrm{e}^{2u} + \text{ constant} \\ \int u\mathrm{e}^{2u}\mathrm{d}u &= \frac{1}{4}(2u-1)\,\mathrm{e}^{2u} + \text{ constant} \\ \int u^2\mathrm{e}^{2u}\mathrm{d}u &= \frac{1}{4}(1-2u+2u^2)\,\mathrm{e}^{2u} + \text{ constant} \end{aligned}$$

$$\begin{aligned} M_y'(\nu) &= \frac{-8\nu}{(4+\nu^2)^2} \\ M_y''(\nu) &= \frac{8(3\nu^2-4)}{(4+\nu^2)^3} \\ M_y^{(3)}(\nu) &= \frac{-96\nu(\nu^2-4)}{(4+\nu^2)^4} \\ M_y^{(4)}(\nu) &= \frac{96(5\nu^4-40\nu^2+16)}{(4+\nu^2)^5} \end{aligned}$$

END

Table of the $Q(\quad)$ and $\text{erf}(\quad)$ functions

The approximation $Q(x) \approx \frac{1}{x\sqrt{2\pi}}(1 - \frac{0.7}{x^2})e^{-x^2/2}$ may be used when $x > 2$.

x	erf(x)	$Q(x)$	x	erf(x)	$Q(x)$	x	erf(x)	$Q(x)$
0.00	0	0.5	1.70	0.9838	0.04457	3.40	1	0.0003369
0.10	0.1125	0.4602	1.80	0.9891	0.03593	3.50	1	0.0002326
0.20	0.2227	0.4207	1.90	0.9928	0.02872	3.60	1	0.0001591
0.30	0.3286	0.3821	2.00	0.9953	0.02275	3.70	1	0.0001078
0.40	0.4284	0.3446	2.10	0.997	0.01786	3.80	1	7.235×10^{-5}
0.50	0.5205	0.3085	2.20	0.9981	0.0139	3.90	1	4.810×10^{-5}
0.60	0.6039	0.2743	2.30	0.9989	0.01072	4.00	1	3.167×10^{-5}
0.70	0.6778	0.242	2.40	0.9993	0.008198	4.10	1	2.066×10^{-5}
0.80	0.7421	0.2119	2.50	0.9996	0.00621	4.20	1	1.335×10^{-5}
0.90	0.7969	0.1841	2.60	0.9998	0.004661	4.30	1	8.540×10^{-6}
1.00	0.8427	0.1587	2.70	0.9999	0.003467	4.40	1	5.413×10^{-6}
1.10	0.8802	0.1357	2.80	0.9999	0.002555	4.50	1	3.398×10^{-6}
1.20	0.9103	0.1151	2.90	1	0.001866	4.60	1	2.112×10^{-6}
1.30	0.934	0.0968	3.00	1	0.00135	4.70	1	1.301×10^{-6}
1.40	0.9523	0.08076	3.10	1	0.0009676	4.80	1	7.933×10^{-7}
1.50	0.9661	0.06681	3.20	1	0.0006871	4.90	1	4.792×10^{-7}
1.60	0.9763	0.0548	3.30	1	0.0004834	5.00	1	2.867×10^{-7}