

**EE501: An Introduction to the Theory of Statistical Communications**

Thursday, 16 October 2014

**Second Quiz**

- REMARKS:
1. Hand held calculator is allowed,
  2. Open book quiz,
  3. Formula sheets are attached,
  4. Marks distribution:
    - Question #1: 6 points
    - Question #2: 4 points
  5. Justify all your answers.

# 1	
# 2	

A communication system is used to transmit a voltage  $s$  to a decision device as shown in Fig. 1. Thus the decision device has available two received voltages,  $r_1$  and  $r_2$ , in which

$$r_1 = s + n_1, \quad r_2 = n_1 + n_2$$

Assume that  $n_1, n_2$  are zero-mean Gaussian random variables with variances  $\sigma_1^2 = \sigma_2^2 = \sigma^2$  and that  $s, n_1$  and  $n_2$  are jointly statistically independent. The system is used to communicate one of two messages  $m_0$  and  $m_1$  with a priori probabilities  $P[m_0] = P[m_1] = 1/2$ . For message  $m_l$  the voltage is

$$s_l = (-1)^l \sqrt{E}; \quad l = 0, 1.$$

The optimum decision rule seeks to determine that  $l$  for which the mixed form joint density function  $p_{r_1, r_2}(\rho_1, \rho_2, \{s = s_l\})$  is maximum.

1. Determine the structure of the optimum decision device.

**Hint:** Start from

$$\begin{aligned} p_{r_1, r_2}(\rho_1, \rho_2, \{s = s_l\}) &= P[m_l] p_{r_1}(\rho_1 | \{s = s_l\}) p_{r_2}(\rho_2 | \{s = s_l\}, r_1 = \rho_1) \\ &= P[m_l] p_{n_1}(\rho_1 - s_l) p_{n_2}(\rho_2 - \rho_1 + s_l) \end{aligned}$$

and show that

$$\hat{m}(\rho_1, \rho_2) = \begin{cases} m_0 & ; \quad 2\rho_1 - \rho_2 > 0 \\ m_1 & ; \quad \text{otherwise} \end{cases}$$

2. Calculate the probability of error.

**Hint:** By symmetry  $P(\mathcal{E}) = P(\mathcal{E} | \{s = s_0\}) = P(\mathcal{E} | \{s = s_1\})$  (you need not show this). Starting from the above optimum decision device, show that  $P(\mathcal{E} | \{s = s_1\})$  reduces to  $P(\{n > 2\sqrt{E}\})$ , where  $n = n_1 - n_2$  is a Gaussian random variable. Calculate the mean and variance of  $n$  and finally express the probability of error in terms of the  $Q(\cdot)$  function.

Name: \_\_\_\_\_

College Number: \_\_\_\_\_

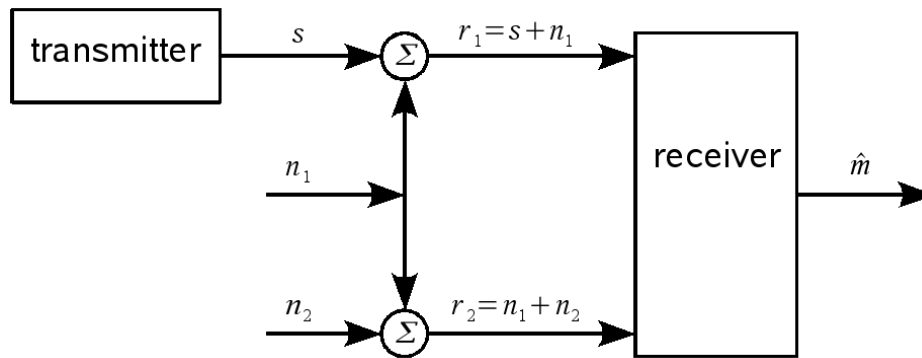


Figure 1:

END

## Formula Sheets

$$\cos \theta = \sin(\theta + 90^\circ)$$

$$2 \sin u \sin v = \cos(u - v) - \cos(u + v)$$

$$2 \sin \theta \cos \theta = \sin(2\theta)$$

$$2 \sin^2 \theta = 1 - \cos(2\theta)$$

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$F_x(\alpha) = P(\{x \leq \alpha\}) = \int_{-\infty}^{\alpha} p_x(u) du$$

$$p_x(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\alpha-\mu)^2/(2\sigma^2)}$$

$$\begin{aligned} \operatorname{erf}(\alpha) &= \frac{2}{\sqrt{\pi}} \int_0^{\alpha} e^{-\beta^2} d\beta \\ &= 1 - 2Q(\sqrt{2}\alpha) \end{aligned}$$

$$P(a < x \leq b) = Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right)$$

$$\frac{dQ(\alpha)}{d\alpha} = \frac{-1}{\sqrt{2\pi}} e^{-\alpha^2/2}$$

$$y = bx + a \Rightarrow p_y(\alpha) = \frac{1}{|b|} p_x\left(\frac{\alpha-a}{b}\right)$$

$$p_{g(x)}(\beta) = p_y(\beta) = \begin{cases} \sum_{\alpha \in S(\beta)} \frac{p_x(\alpha)}{|g'(\alpha)|} & ; \text{ if } S(\beta) \neq \emptyset \text{ and} \\ & g'(\alpha) \neq 0, \forall \alpha \in S(\beta) \triangleq \{\alpha \in \mathbb{R} : \beta = g(\alpha)\} \\ 0 & ; \text{ if } S(y) = \emptyset \end{cases}$$

$$p_{f(x)}(\beta) = p_y(\beta) = p_x(g(\beta)) |J_g(\beta)|$$

$$2 \cos u \cos v = \cos(u - v) + \cos(u + v)$$

$$2 \sin u \cos v = \sin(u - v) + \sin(u + v)$$

$$2 \cos^2 \theta = 1 + \cos(2\theta)$$

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$P(AB) = P(A|B)P(B) = P(B|A)P(A)$$

$$p_x(\alpha) = \frac{dF_x(\alpha)}{d\alpha}$$

$$P(a < x \leq b) = \int_a^b p_x(\alpha) d\alpha$$

$$\begin{aligned} Q(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\beta^2/2} d\beta \\ &= \frac{1}{2} \left[ 1 - \operatorname{erf}\left(\frac{\alpha}{\sqrt{2}}\right) \right] \\ &= 1 - Q(-\alpha) \end{aligned}$$

$$P(x > a) = Q\left(\frac{a-\mu}{\sigma}\right)$$

$$P(x \leq a) = Q\left(\frac{\mu-a}{\sigma}\right)$$

$$p_x(\alpha) = \int_{-\infty}^{\infty} p_{xy}(\alpha, \beta) d\beta$$

$$p_x(\alpha|y = v) = p_{x|y}(\alpha, v) = \frac{p_{xy}(\alpha, v)}{p_y(v)}$$

$$x, y \text{ are independent} \Leftrightarrow p_{xy}(\alpha, \beta) = p_x(\alpha)p_y(\beta)$$

## Formula Sheets (continued)

### Fourier Transform Properties

Operation	$g(t)$	$G(f)$
Addition	$g_1(t) + g_2(t)$	$G_1(f) + G_2(f)$
Multiplication by a constant	$ag(t)$	$aG(f)$
Symmetry	$G(t)$	$g(-f)$
Scaling	$g(at)$	$\frac{1}{ a }G\left(\frac{f}{a}\right)$
Time shifting	$g(t - t_0)$	$e^{-j2\pi ft_0}G(f)$
Frequency Shifting	$e^{j2\pi f_0 t}g(t)$	$G(f - f_0)$
Modulation	$2g(t) \cos(2\pi f_c t)$	$G(f - f_c) + G(f + f_c)$
Time Differentiation	$\frac{d^k g(t)}{dt^k}$	$(j2\pi f)^k G(f)$
Frequency Differentiation	$(-j2\pi t)^n g(t)$	$\frac{d^n G(f)}{df^n}$
Complex Conjugate	$g^*(t)$	$G^*(-f)$
Time Domain Convolution	$g_1(t) * g_2(t)$	$G_1(f)G_2(f)$
Time Domain Multiplication	$g_1(t)g_2(t)$	$G_1(f) * G_2(f)$
Parseval Theorem	$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)dt$	$\int_{-\infty}^{\infty} G_1(f)G_2^*(f)df$
Time Domain Integration	$\int_{-\infty}^t g(x)dx$	$\frac{G(f)}{j2\pi f} + \frac{G(0)}{2}\delta(f)$

## Formula Sheets (continued)

### Table of $Q(\cdot)$ and $\text{erf}(\cdot)$ functions

The approximation  $Q(x) \approx \frac{1}{x\sqrt{2\pi}}(1 - \frac{0.7}{x^2})e^{-x^2/2}$  may be used when  $x > 2$ .

$x$	$\text{erf}(x)$	$Q(x)$	$x$	$\text{erf}(x)$	$Q(x)$	$x$	$\text{erf}(x)$	$Q(x)$
0.00	0	0.5	1.70	0.9838	0.04457	3.40	1	0.0003369
0.10	0.1125	0.4602	1.80	0.9891	0.03593	3.50	1	0.0002326
0.20	0.2227	0.4207	1.90	0.9928	0.02872	3.60	1	0.0001591
0.30	0.3286	0.3821	2.00	0.9953	0.02275	3.70	1	0.0001078
0.40	0.4284	0.3446	2.10	0.997	0.01786	3.80	1	$7.235 \times 10^{-5}$
0.50	0.5205	0.3085	2.20	0.9981	0.0139	3.90	1	$4.810 \times 10^{-5}$
0.60	0.6039	0.2743	2.30	0.9989	0.01072	4.00	1	$3.167 \times 10^{-5}$
0.70	0.6778	0.242	2.40	0.9993	0.008198	4.10	1	$2.066 \times 10^{-5}$
0.80	0.7421	0.2119	2.50	0.9996	0.00621	4.20	1	$1.335 \times 10^{-5}$
0.90	0.7969	0.1841	2.60	0.9998	0.004661	4.30	1	$8.540 \times 10^{-6}$
1.00	0.8427	0.1587	2.70	0.9999	0.003467	4.40	1	$5.413 \times 10^{-6}$
1.10	0.8802	0.1357	2.80	0.9999	0.002555	4.50	1	$3.398 \times 10^{-6}$
1.20	0.9103	0.1151	2.90	1	0.001866	4.60	1	$2.112 \times 10^{-6}$
1.30	0.934	0.0968	3.00	1	0.00135	4.70	1	$1.301 \times 10^{-6}$
1.40	0.9523	0.08076	3.10	1	0.0009676	4.80	1	$7.933 \times 10^{-7}$
1.50	0.9661	0.06681	3.20	1	0.0006871	4.90	1	$4.792 \times 10^{-7}$
1.60	0.9763	0.0548	3.30	1	0.0004834	5.00	1	$2.867 \times 10^{-7}$