

Name: _____

College Number: _____

EE501: An Introduction to the Theory of Statistical Communications

Tuesday, 29 September 2015

First Quiz

- REMARKS:
1. Hand held calculator is allowed,
 2. Open book quiz,
 3. Formula sheets are attached,
 4. Marks distribution:
Question #1: 3 points
Question #2: 4 points
Question #3: 3 points
 5. Justify all your answers.

# 1	
# 2	
# 3	

1. A probability system $(\Omega, \mathcal{F}, P(\cdot))$ is defined by:

- $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$,
- $\mathcal{F} = 2^\Omega$,
- $P(\cdot)$ is defined by:

$$P : \mathcal{F} \rightarrow [0, 1]$$
$$P : E \mapsto \frac{|E|}{8}$$

or, in other words, the events consisting of a single sample point are equiprobable.

Consider the following events $A, B, C \in \mathcal{F}$:

$$A = \{1, 2, 3, 4\}$$

$$B = \{2, 3, 5, 6\}$$

$$C = \{3, 4, 6, 7\}$$

- (a) Are A, B, C statistically independent?
 - (b) Are the events $E = \overline{A} \cup B$ and $F = \overline{C}$ statistically independent?
2. A symbol from the two-symbol alphabet $\{m_0, m_1\}$ is transmitted over a noisy discrete communication channel and a symbol from the three-symbol alphabet $\{r_0, r_1, r_2\}$ is received. The transition probabilities of the channel are specified by the diagram of figure 1. A source is available that uses m_0 and m_1 with the following probabilities:

$$P(m_0) = 0.6,$$

$$P(m_1) = 0.4.$$

What is the best receiver decision rule (assignment of r_0, r_1, r_2 to m_0, m_1) and what is the resulting probability of error?

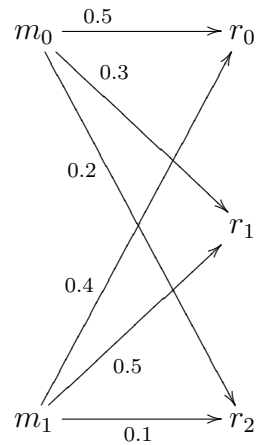


Figure 1:

3. Consider the probability space defined by:¹

- the sample space is the real-line interval $\Omega = \{\omega : 0 \leq \omega \leq 1\}$;
- the class of events is the set of all intervals of this line segment, plus (countable) unions, (countable) intersections, and complements of such intervals. The intervals may include both, one, or either of the end points;
- the probability assignment is the sum of the lengths of the disjoint intervals that constitute the event.

Calculate the probability distribution function $F_x(\alpha)$ of the random variable x defined by:

$$\begin{aligned} x : \Omega &\rightarrow \mathbb{R} \\ x : \omega &\mapsto \omega^2 \end{aligned}$$

Note: You need not show that x is a *valid* random variable.

¹W&J, bottom of page 21 and top of page 22.

END

Formula Sheets

$$\cos \theta = \sin(\theta + 90^\circ)$$

$$2 \sin u \sin v = \cos(u - v) - \cos(u + v)$$

$$2 \sin \theta \cos \theta = \sin(2\theta)$$

$$2 \sin^2 \theta = 1 - \cos(2\theta)$$

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$F_x(\alpha) = P(\{x \leq \alpha\}) = \int_{-\infty}^{\alpha} p_x(u) du$$

$$p_x(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\alpha-\mu)^2/(2\sigma^2)}$$

$$\begin{aligned} \operatorname{erf}(\alpha) &= \frac{2}{\sqrt{\pi}} \int_0^{\alpha} e^{-\beta^2} d\beta \\ &= 1 - 2Q(\sqrt{2}\alpha) \end{aligned}$$

$$P(a < x \leq b) = Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right)$$

$$\frac{dQ(\alpha)}{d\alpha} = \frac{-1}{\sqrt{2\pi}} e^{-\alpha^2/2}$$

$$y = bx + a \Rightarrow p_y(\alpha) = \frac{1}{|b|} p_x\left(\frac{\alpha-a}{b}\right)$$

$$p_{g(x)}(\beta) = p_y(\beta) = \begin{cases} \sum_{\alpha \in S(\beta)} \frac{p_x(\alpha)}{|g'(\alpha)|} & ; \text{ if } S(\beta) \neq \emptyset \text{ and} \\ & g'(\alpha) \neq 0, \forall \alpha \in S(\beta) \triangleq \{\alpha \in \mathbb{R} : \beta = g(\alpha)\} \\ 0 & ; \text{ if } S(\beta) = \emptyset \end{cases}$$

$$p_{f(x)}(\beta) = p_y(\beta) = p_x(g(\beta)) |J_g(\beta)|$$

$$2 \cos u \cos v = \cos(u - v) + \cos(u + v)$$

$$2 \sin u \cos v = \sin(u - v) + \sin(u + v)$$

$$2 \cos^2 \theta = 1 + \cos(2\theta)$$

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$P(AB) = P(A|B)P(B) = P(B|A)P(A)$$

$$p_x(\alpha) = \frac{dF_x(\alpha)}{d\alpha}$$

$$P(a < x \leq b) = \int_a^b p_x(\alpha) d\alpha$$

$$\begin{aligned} Q(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\beta^2/2} d\beta \\ &= \frac{1}{2} \left[1 - \operatorname{erf}\left(\frac{\alpha}{\sqrt{2}}\right) \right] \\ &= 1 - Q(-\alpha) \end{aligned}$$

$$P(x > a) = Q\left(\frac{a-\mu}{\sigma}\right)$$

$$P(x \leq a) = Q\left(\frac{\mu-a}{\sigma}\right)$$

$$p_x(\alpha) = \int_{-\infty}^{\infty} p_{xy}(\alpha, \beta) d\beta$$

$$p_x(\alpha|y = v) = p_{x|y}(\alpha, v) = \frac{p_{xy}(\alpha, v)}{p_y(v)}$$

$$x, y \text{ are independent} \Leftrightarrow p_{xy}(\alpha, \beta) = p_x(\alpha)p_y(\beta)$$

Formula Sheets (continued)

Fourier Transform Properties

Operation	$g(t)$	$G(f)$
Addition	$g_1(t) + g_2(t)$	$G_1(f) + G_2(f)$
Multiplication by a constant	$ag(t)$	$aG(f)$
Symmetry	$G(t)$	$g(-f)$
Scaling	$g(at)$	$\frac{1}{ a }G\left(\frac{f}{a}\right)$
Time shifting	$g(t - t_0)$	$e^{-j2\pi ft_0}G(f)$
Frequency Shifting	$e^{j2\pi f_0 t}g(t)$	$G(f - f_0)$
Modulation	$2g(t) \cos(2\pi f_c t)$	$G(f - f_c) + G(f + f_c)$
Time Differentiation	$\frac{d^k g(t)}{dt^k}$	$(j2\pi f)^k G(f)$
Frequency Differentiation	$(-j2\pi t)^n g(t)$	$\frac{d^n G(f)}{df^n}$
Complex Conjugate	$g^*(t)$	$G^*(-f)$
Time Domain Convolution	$g_1(t) * g_2(t)$	$G_1(f)G_2(f)$
Time Domain Multiplication	$g_1(t)g_2(t)$	$G_1(f) * G_2(f)$
Parseval Theorem	$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)dt$	$\int_{-\infty}^{\infty} G_1(f)G_2^*(f)df$
Time Domain Integration	$\int_{-\infty}^t g(x)dx$	$\frac{G(f)}{j2\pi f} + \frac{G(0)}{2}\delta(f)$

Formula Sheets (continued)

Table of $Q(\cdot)$ and $\text{erf}(\cdot)$ functions

The approximation $Q(x) \approx \frac{1}{x\sqrt{2\pi}}(1 - \frac{0.7}{x^2})e^{-x^2/2}$ may be used when $x > 2$.

x	$\text{erf}(x)$	$Q(x)$	x	$\text{erf}(x)$	$Q(x)$	x	$\text{erf}(x)$	$Q(x)$
0.00	0	0.5	1.70	0.9838	0.04457	3.40	1	0.0003369
0.10	0.1125	0.4602	1.80	0.9891	0.03593	3.50	1	0.0002326
0.20	0.2227	0.4207	1.90	0.9928	0.02872	3.60	1	0.0001591
0.30	0.3286	0.3821	2.00	0.9953	0.02275	3.70	1	0.0001078
0.40	0.4284	0.3446	2.10	0.997	0.01786	3.80	1	7.235×10^{-5}
0.50	0.5205	0.3085	2.20	0.9981	0.0139	3.90	1	4.810×10^{-5}
0.60	0.6039	0.2743	2.30	0.9989	0.01072	4.00	1	3.167×10^{-5}
0.70	0.6778	0.242	2.40	0.9993	0.008198	4.10	1	2.066×10^{-5}
0.80	0.7421	0.2119	2.50	0.9996	0.00621	4.20	1	1.335×10^{-5}
0.90	0.7969	0.1841	2.60	0.9998	0.004661	4.30	1	8.540×10^{-6}
1.00	0.8427	0.1587	2.70	0.9999	0.003467	4.40	1	5.413×10^{-6}
1.10	0.8802	0.1357	2.80	0.9999	0.002555	4.50	1	3.398×10^{-6}
1.20	0.9103	0.1151	2.90	1	0.001866	4.60	1	2.112×10^{-6}
1.30	0.934	0.0968	3.00	1	0.00135	4.70	1	1.301×10^{-6}
1.40	0.9523	0.08076	3.10	1	0.0009676	4.80	1	7.933×10^{-7}
1.50	0.9661	0.06681	3.20	1	0.0006871	4.90	1	4.792×10^{-7}
1.60	0.9763	0.0548	3.30	1	0.0004834	5.00	1	2.867×10^{-7}