ROYAL MILITARY COLLEGE OF CANADA

FALL TERM EXAMINATIONS 2018 – 2019

PG FINAL EXAMINATION

EE501: INTRODUCTION TO THE THEORY OF STATISTICAL COMMUNICATION

Monday, 17 December 2018, 1300 - 1600 h.

EXAMINER: G Drolet, Associate Professor

TIME: 3 hours

NOTES:

1. Hand held calculator is allowed.
2. Only the textbook (Wozencraft & Jacobs) and course notes are allowed; problem solutions are not allowed.
3. Answer only two of questions 1, 2, 3. Answer all of question 4.
4. The questions have the following value:
   Question #1, 5 points
   Question #2, 5 points
   Question #3: 5 points
   Question #4: 15 points
5. Justify all your answers.

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EE501: INTRODUCTION TO THE THEORY OF STATISTICAL COMMUNICATION
Question 1

Answer only two of questions 1, 2, 3.

1. [1] A communication system is used to transmit one of two equally likely messages, m_0 and m_1. The channel output is a continuous random variable r, the conditional density functions of which are shown in Fig. 1. Determine the optimum receiver decision rule and compute the resulting probability of error.

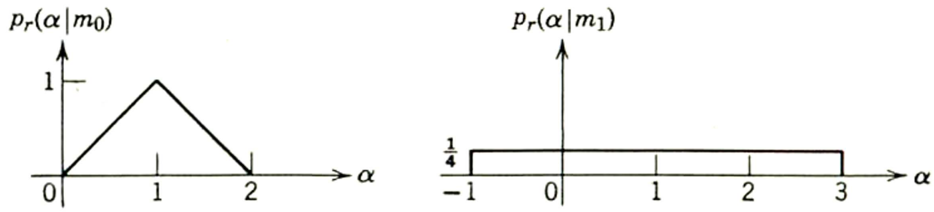

Figure 1:

2. [2] It is known that $\mathrm{P}[\mathscr{E}]_{\min} = q$ when the two signal vectors $\boldsymbol{s}_0$ and $\boldsymbol{s}_1$ shown in fig $2a$ are transmitted with equal probability over a channel disturbed by additive white Gaussian noise. Compute $\mathrm{P}[\mathscr{E}]_{\min}$ in terms of q, θ, and l when the nine vectors indicated by $\times$'s in fig $2b$ are used as signals with equal probability over the same channel.

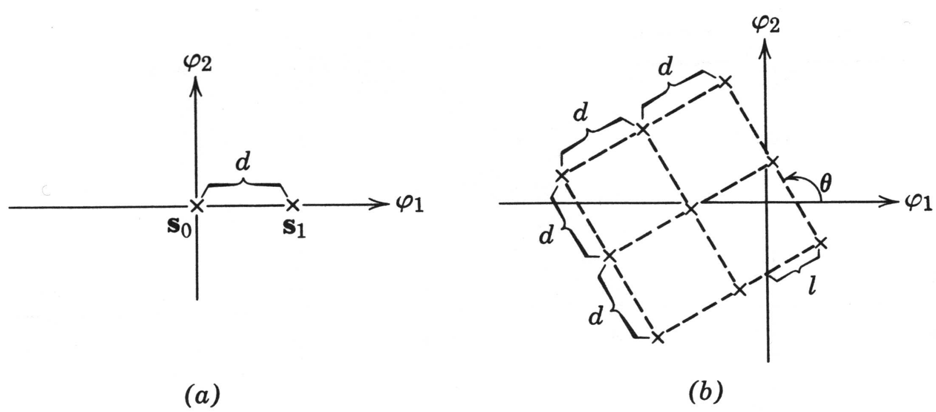

Figure 2:

[1]Problem 2.23 in the textbook by Wozencraft & Jacobs.
[2]Problem 4.3, pages 274 in W & J.

3. In the diagram of figure 3, $n_w(t)$ denotes a white Gaussian noise with power spectral density $S_w(f) = \frac{\mathcal{N}_0}{2}$ and $r = y(0)$. Express $\overline{r}$, $Var(r)$ as a function of $\mathcal{N}_0/2$.
   Hint: You may interchange $E[\quad]$ and $\int$ and we recall that:

$$\begin{aligned} f(t) * g(t) &= g(t) * f(t) \\ &= \int_{-\infty}^{\infty} f(\alpha) g(t-\alpha) \mathrm{d}\alpha \\ &= \int_{-\infty}^{\infty} g(\alpha) f(t-\alpha) \mathrm{d}\alpha \end{aligned}$$

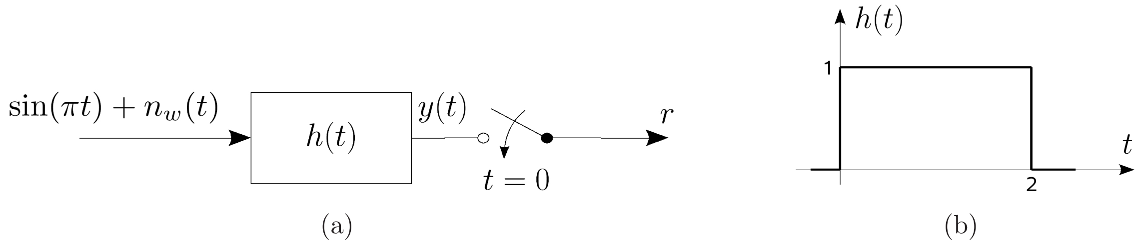

Figure 3:

Answer all of question 4.

4. The three signals $s_0(t)$, $s_1(t)$, $s_2(t)$ given by[3]:

$$\begin{aligned} s_0(t) &= (u(t) - u(t-1))/2 \\ s_1(t) &= t\,(u(t) - u(t-1)) \\ s_2(t) &= \cos(2\pi t)\,(u(t) - u(t-1)) \end{aligned}$$

   are used to communicate one of three equally likely messages on a channel with additive white Gaussian noise $n_w(t)$ statistically independent of the message and with power spectral density $\mathcal{N}_0/2$. $s_0(t)$, $s_1(t)$, $s_2(t)$ are sketched in figure 4 and their energy are easily found (you need not verify this):

$$\begin{aligned} E_{s_0} &= 1/4 \\ E_{s_1} &= 1/3 \\ E_{s_2} &= 1/2 \end{aligned}$$

   The following is an orthonormal basis suitable for the representation of signals $s_0(t)$, $s_1(t)$ (you need not verify this):

$$\begin{aligned} \phi_1(t) &= \sqrt{3}(1-t)\,(u(t) - u(t-1)) \\ \phi_2(t) &= (3t-1)\,(u(t) - u(t-1)) \end{aligned}$$

   $\phi_1(t)$, $\phi_2(t)$ are sketched in figure 5.

[3] $u(t)$ denotes as usual the unit step function.

Question 4

(a) Calculate a basis function $\phi_3(t)$, if required, which, together with $\phi_1(t)$, $\phi_2(t)$, forms an orthonormal basis suitable for the representation of all three signals $s_0(t)$, $s_1(t)$ and $s_2(t)$.
Note: If you are unable to calculate $\phi_3(t)$, you may use the basis function $\phi_3'(t)$ given by:

$$\phi_3'(t) = \sqrt{\frac{5\pi^2}{180 + 91\pi^2}}\,\left(6\,t^2 + 6\,\cos(2\,\pi\,t) - 6\,t + 1\right)$$

in the subquestions that follow. $\{\phi_1(t),\, \phi_2(t),\, \phi_3'(t)\}$ forms an orthonormal basis that spans $s_0(t)$, $s_1(t)$ but not $s_2(t)$; it may nonetheless be used in the remainder of this problem.

(b) Find vector representation of signals $s_0(t)$, $s_1(t)$, $s_2(t)$ in the basis $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$ (i.e. as triplets).
Hint: Some reflection may save you a lot of time; think before you start some long and tedious calculations.
Note: If you are using $\phi_3'(t)$, then you would find:

$$\langle s_2(t),\, \phi_3'(t)\rangle = \frac{3(1+\pi^2)}{\pi}\sqrt{\frac{5}{180 + 91\pi^2}} \approx 0.7068605607$$

Do not calculate this value.

(c) If the transmitted waveform and the noise are such that the received waveform is

$$r(t) = s(t) + n_w(t) = \frac{4}{5}\,(10t^2 - 10t + 1)$$

which of the three signals $s_0(t)$, $s_1(t)$, $s_2(t)$ was most likely transmitted.
Hint: Again some reflection may save you a lot of time; choose carefully which receiver implementation to use.

(d) What is the average energy per message and what value of $\mathscr{N}_0$ corresponds to a signal-to-noise ratio of 15 dB.
Note: If you can't calculate the value of $\mathscr{N}_0$ you may use $\mathscr{N}_0 = 0.01$ in the next subquestion.

(e) Use the union bound to obtain an upper bound on the probability of error of the communication system at a signal to noise ratio of 15 dB.

The following may be useful:

$$\sqrt{\langle \boldsymbol{u}-\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\rangle} \equiv \text{distance between } \boldsymbol{u}, \boldsymbol{v}$$

$$\int t^n \mathrm{d}t = \frac{t^{n+1}}{n+1}$$

$$\int \frac{1}{t} \mathrm{d}t = \ln(t)$$

$$\int \cos(2\pi t)\,\mathrm{d}t = \frac{\sin(2\pi t)}{2\pi}$$

$$\int \cos^2(2\pi t)\,\mathrm{d}t = \frac{\cos(2\pi t)\sin(2\pi t) + 2\pi t}{4\pi}$$

$$\int t\cos(2\pi t)\,\mathrm{d}t = \frac{\cos(2\pi t) + 2t\sin(2\pi t)\pi}{4\pi^2}$$

$$\int t^2\cos(2\pi t)\,\mathrm{d}t = \frac{2\pi^2 t^2\sin(2\pi t) - \sin(2\pi t) + 2\pi t\cos(2\pi t)}{4\pi^3}$$

$$\int t^3\cos(2\pi t)\,\mathrm{d}t = \frac{4\pi^3 t^3\sin(2\pi t) + 6\pi^2 t^2\cos(2\pi t) - 3\cos(2\pi t) - 6t\sin(2\pi t)\pi}{8\pi^4}$$

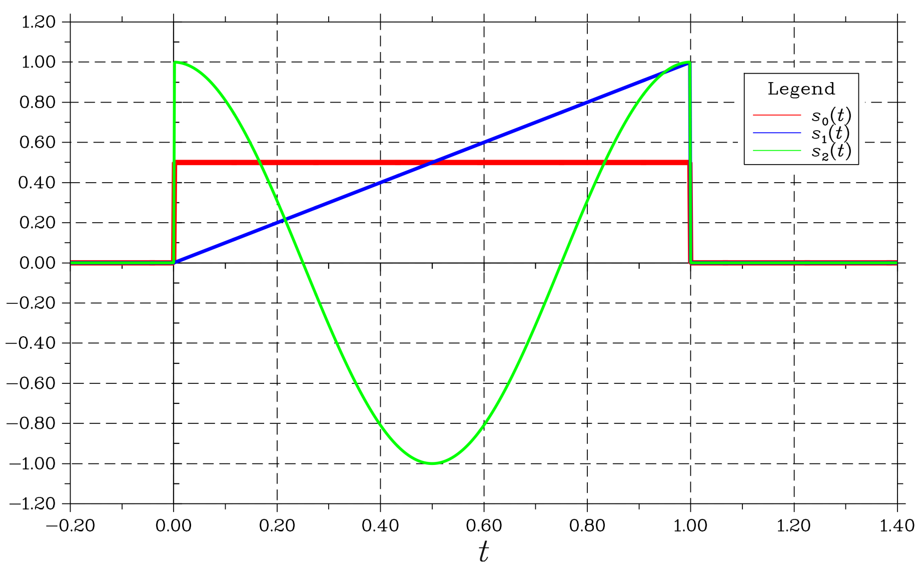

Figure 4:

Question 4

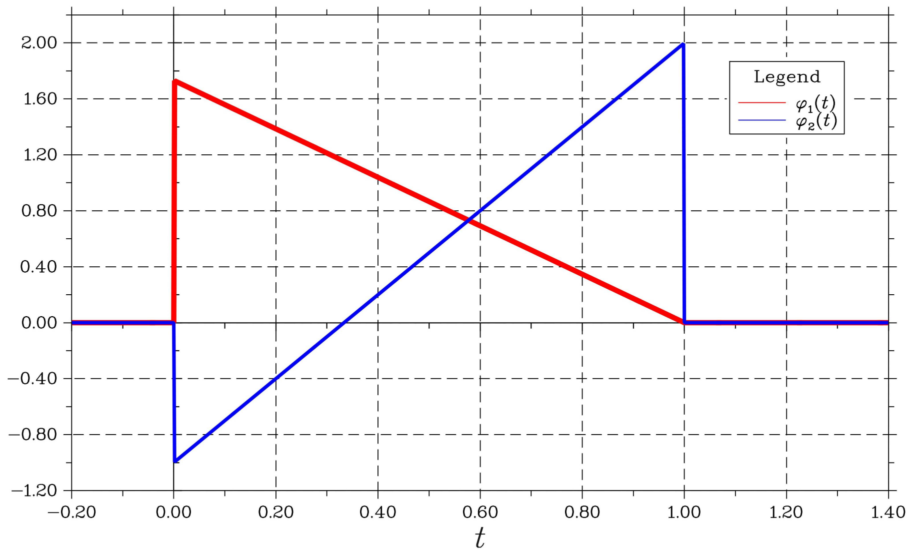

Figure 5:

END

Table of the $Q(\quad)$ and $\mathrm{erf}(\quad)$ functions

The approximation $Q(x) \approx \frac{1}{x\sqrt{2\pi}}(1 - \frac{0.7}{x^2})e^{-x^2/2}$ may be used when $x > 2$.

x	erf(x)	$Q(x)$	x	erf(x)	$Q(x)$	x	erf(x)	$Q(x)$
0.00	0	0.5	1.70	0.9838	0.04457	3.40	1	0.0003369
0.10	0.1125	0.4602	1.80	0.9891	0.03593	3.50	1	0.0002326
0.20	0.2227	0.4207	1.90	0.9928	0.02872	3.60	1	0.0001591
0.30	0.3286	0.3821	2.00	0.9953	0.02275	3.70	1	0.0001078
0.40	0.4284	0.3446	2.10	0.997	0.01786	3.80	1	7.235×10^{-5}
0.50	0.5205	0.3085	2.20	0.9981	0.0139	3.90	1	4.810×10^{-5}
0.60	0.6039	0.2743	2.30	0.9989	0.01072	4.00	1	3.167×10^{-5}
0.70	0.6778	0.242	2.40	0.9993	0.008198	4.10	1	2.066×10^{-5}
0.80	0.7421	0.2119	2.50	0.9996	0.00621	4.20	1	1.335×10^{-5}
0.90	0.7969	0.1841	2.60	0.9998	0.004661	4.30	1	8.540×10^{-6}
1.00	0.8427	0.1587	2.70	0.9999	0.003467	4.40	1	5.413×10^{-6}
1.10	0.8802	0.1357	2.80	0.9999	0.002555	4.50	1	3.398×10^{-6}
1.20	0.9103	0.1151	2.90	1	0.001866	4.60	1	2.112×10^{-6}
1.30	0.934	0.0968	3.00	1	0.00135	4.70	1	1.301×10^{-6}
1.40	0.9523	0.08076	3.10	1	0.0009676	4.80	1	7.933×10^{-7}
1.50	0.9661	0.06681	3.20	1	0.0006871	4.90	1	4.792×10^{-7}
1.60	0.9763	0.0548	3.30	1	0.0004834	5.00	1	2.867×10^{-7}