ROYAL MILITARY COLLEGE OF CANADA

FALL TERM EXAMINATIONS 2015 – 2016

PG FINAL EXAMINATION

EE501: INTRODUCTION TO THE THEORY OF STATISTICAL COMMUNICATION

Thursday, 10 December 2015, 1300 - 1600 h.

EXAMINER: G Drolet, Associate Professor

TIME: 3 hours

NOTES:

1. Hand held calculator is allowed.
2. Only the textbook (Wozencraft & Jacobs) and course notes are allowed; problem solutions are not allowed.
3. Answer all the questions.
4. The questions all have the same value.
5. Formula sheets are attached.
6. Justify all your answers.

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EE501: INTRODUCTION TO THE THEORY OF STATISTICAL COMMUNICATION

Question 1

1. In a ternary communication system, a random message m is transmitted, where m is one of three equally likely symbols m_0, m_1 or m_2: $P(m_0) = P(m_1) = P(m_2) = \frac{1}{3}$. The received random variable is denoted by r and from a received value $r = \rho$, a decision denoted by $\widehat{m}(\rho)$ is made: $\widehat{m}(\rho) \in \{m_0, m_1, m_2\}$. As usual we let $\mathscr{E}$ denote the *error event*, i.e. $\widehat{m}(r) \neq m$ and we let $\mathscr{C}$ denote the *correct decision event*, i.e. $\widehat{m}(r) = m$. The conditional probability density functions of the received random variable r given m are sketched on figure 1.

   (a) If $r = 1.5$ is received, which of the three messages was most likely transmitted.

   (b) The following decision rule (not necessarily the best) is used in parts (1b) and (1c):

$$\widehat{m}(\rho) = \begin{cases} m_0 & ; \quad \rho < 1.25 \\ m_1 & ; \quad 1.25 < \rho < 4.375 \\ m_2 & ; \quad \rho > 4.375 \end{cases} \tag{1}$$

   Calculate $P(\widehat{m}(r) = m_0)$, $P(\widehat{m}(r) = m_1)$, $P(\widehat{m}(r) = m_2)$.

   (c) Calculate $P(\mathscr{E})$ with the decision rule (1).
   Hint: It is easier to first calculate $P(\mathscr{C})$.

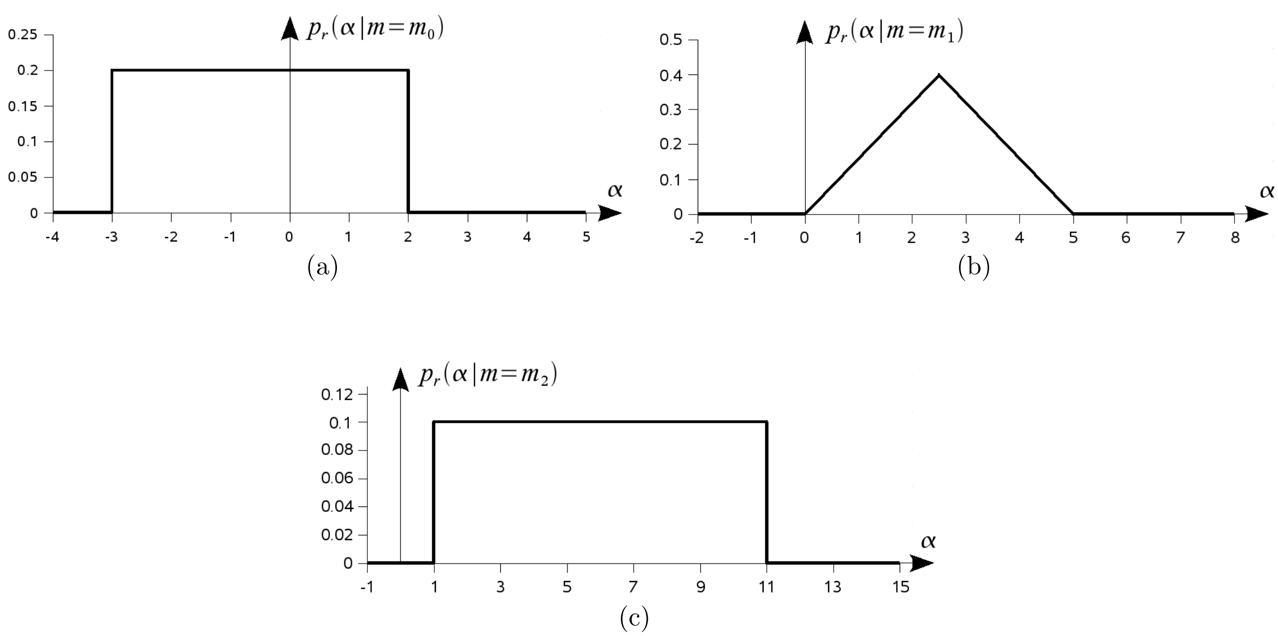

Figure 1:

2. A stationary Gaussian process $x(t)$ with mean function $m_x = 5$ and power spectral density

$$S_x(f) = \begin{cases} 2 & ; \quad -4 < f < -3 \\ 2 + 25\,\delta(f) & ; \quad -1 < f < 1 \\ 2 & ; \quad 3 < f < 4 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

is fed through two linear invariant filters as shown in figure 2. The filters frequency responses are:

$$H_z(f) = \begin{cases} 1 & ; \quad |f| > 1.5 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

$$H_y(f) = \begin{cases} 1 - |f|/2 & ; \quad |f| < 2 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

(a) Calculate m_y, m_z.
Hint: The impulse responses $h_y(t)$, $h_z(t)$ are not required; m_y and m_z can be obtained directly with the frequency responses $H_y(f)$, $H_z(f)$.

(b) Sketch $S_y(f)$, $S_z(f)$, $S_{yz}(f)$.

(c) Are the processes $y(t)$, $z(t)$ statistically independent? Justify your answer.

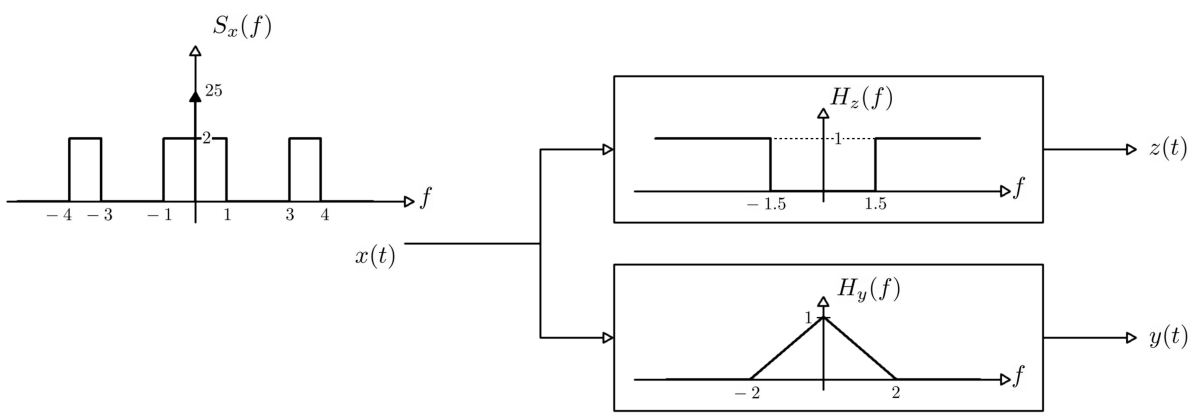

Figure 2:

3. In the diagram of figure 3, $n_w(t)$ denotes a white Gaussian noise with power spectral density $S_w(f) = \frac{\mathcal{N}_0}{2}$ and $r = y(0)$. Express $\overline{r}$, $Var(r)$ as a function of $\mathcal{N}_0/2$.
Hint: You may interchange $E[\quad]$ and $\int$ and we recall that:

$$\begin{aligned} f(t) * g(t) &= g(t) * f(t) \\ &= \int_{-\infty}^{\infty} f(\alpha)g(t-\alpha)\mathrm{d}\alpha \\ &= \int_{-\infty}^{\infty} g(\alpha)f(t-\alpha)\mathrm{d}\alpha \end{aligned}$$

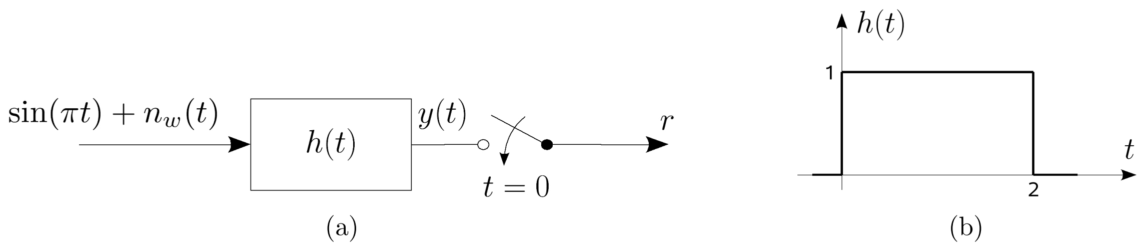

Figure 3:

4. [1] **Suggestion:** Use the *Parseval Theorem* property of the Fourier transform for the signal set in subfigure *(c)* below.

   (a) Calculate $\mathrm{P}[\mathscr{E}]_{\min}$ when the signal sets specified by figs. 4*(a)*, *(b)*, and *(c)* are used to communicate one of two equally likely messages over a channel disturbed by additive Gaussian noise with $S_n(f) = 0.15$.

   (b) Repeat part (a) for a priori message probabilities $\mathrm{P}(m_0) = \frac{1}{4}$, $\mathrm{P}(m_1) = \frac{3}{4}$.

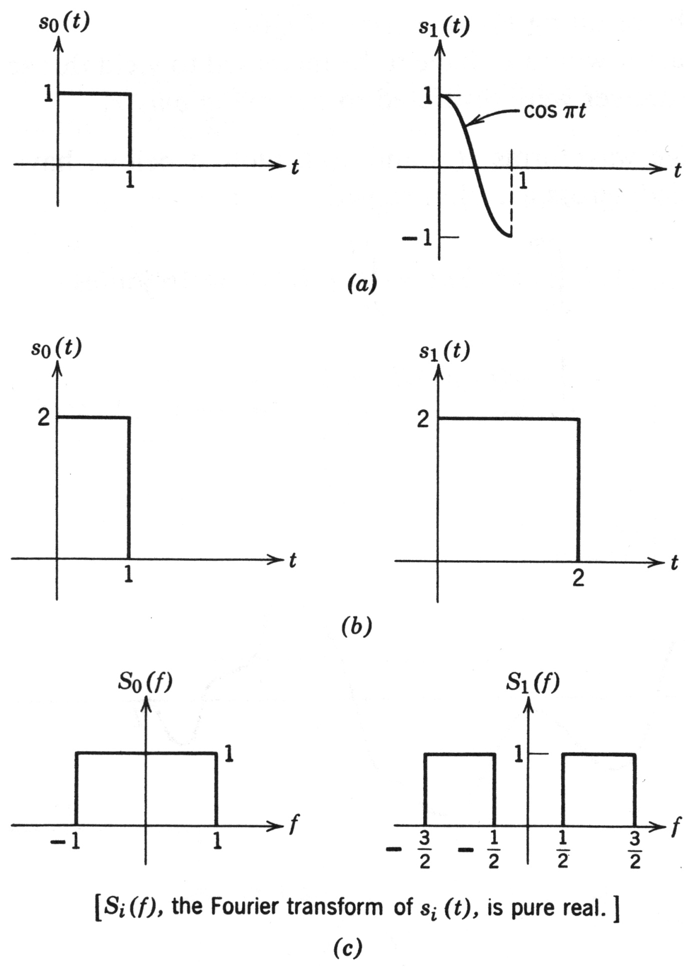

Figure 4:

END

[1]Problem 4.8, pages 278 in W & J.

Formula Sheets

$\cos\theta = \sin(\theta + 90^\circ)$

$2\cos u \cos v = \cos(u-v) + \cos(u+v)$

$2\sin u \sin v = \cos(u-v) - \cos(u+v)$

$2\sin u \cos v = \sin(u-v) + \sin(u+v)$

$2\sin\theta\cos\theta = \sin(2\theta)$

$2\cos^2\theta = 1 + \cos(2\theta)$

$2\sin^2\theta = 1 - \cos(2\theta)$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

$P(AB) = P(A|B)P(B) = P(B|A)P(A)$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$p_x(\alpha) = \frac{\mathrm{d}F_x(\alpha)}{\mathrm{d}\alpha}$

$F_x(\alpha) = P(\{x \le \alpha\}) = \int_{-\infty}^{\alpha} p_x(u)\,\mathrm{d}u$

$P(a < x \le b) = \int_a^b p_x(\alpha)\mathrm{d}\alpha$

$p_x(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\alpha-\mu)^2/(2\sigma^2)}$

$$\begin{aligned} Q(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_\alpha^\infty e^{-\beta^2/2}\mathrm{d}\beta \\ &= \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{\alpha}{\sqrt{2}}\right)\right] \\ &= 1 - Q(-\alpha) \end{aligned}$$

$$\begin{aligned} \operatorname{erf}(\alpha) &= \frac{2}{\sqrt{\pi}}\int_0^\alpha e^{-\beta^2}\mathrm{d}\beta \\ &= 1 - 2Q(\sqrt{2}\alpha) \end{aligned}$$

$P(x > a) = Q\left(\frac{a-\mu}{\sigma}\right)$

$P(a < x \le b) = Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right)$

$P(x \le a) = Q(\frac{\mu-a}{\sigma})$

$\frac{\mathrm{d}Q(\alpha)}{\mathrm{d}\alpha} = \frac{-1}{\sqrt{2\pi}} e^{-\alpha^2/2}$

$p_x(\alpha) = \int_{-\infty}^{\infty} p_{xy}(\alpha,\beta)\mathrm{d}\beta$

$y = bx + a \Rightarrow p_y(\alpha) = \frac{1}{|b|} p_x(\frac{\alpha-a}{b})$

$$p_{g(x)}(\beta) = p_y(\beta) = \begin{cases} \sum_{\alpha\in S(\beta)} \frac{p_x(\alpha)}{|g'(\alpha)|} & ;\ \text{if } S(\beta) \ne \emptyset \text{ and} \\ & \quad g'(\alpha) \ne 0, \forall\alpha \in S(\beta) \triangleq \{\alpha \in \mathbb{R} : \beta = g(\alpha)\} \\ 0 & ;\ \text{if } S(y) = \emptyset \end{cases}$$

$p_{\boldsymbol{f}(\boldsymbol{x})}(\boldsymbol{\beta}) = p_{\boldsymbol{y}}(\boldsymbol{\beta}) = p_{\boldsymbol{x}}(\boldsymbol{g}(\boldsymbol{\beta}))|J_{\boldsymbol{g}}(\boldsymbol{\beta})|$

$p_x(\alpha|y=v) = p_{x|y}(\alpha, v) = \frac{p_{xy}(\alpha,v)}{p_y(v)}$

x, y are independent $\Leftrightarrow p_{xy}(\alpha,\beta) = p_x(\alpha)p_y(\beta)$

Formula Sheets (continued)

Fourier Transform Properties

Operation	$g(t)$	$G(f)$
Addition	$g_1(t) + g_2(t)$	$G_1(f) + G_2(f)$
Multiplication by a constant	$ag(t)$	$aG(f)$
Symmetry	$G(t)$	$g(-f)$
Scaling	$g(at)$	$\frac{1}{\|a\|}G(\frac{f}{a})$
Time shifting	$g(t - t_0)$	$e^{-j2\pi f t_0}G(f)$
Frequency Shifting	$e^{j2\pi f_0 t}g(t)$	$G(f - f_0)$
Modulation	$2g(t)\cos(2\pi f_c t)$	$G(f - f_c) + G(f + f_c)$
Time Differentiation	$\frac{\mathrm{d}^k g(t)}{\mathrm{d}t^k}$	$(j2\pi f)^k G(f)$
Frequency Differentiation	$(-j2\pi t)^n g(t)$	$\frac{\mathrm{d}^n G(f)}{\mathrm{d}f^n}$
Complex Conjugate	$g^*(t)$	$G^*(-f)$
Time Domain Convolution	$g_1(t) * g_2(t)$	$G_1(f)G_2(f)$
Time Domain Multiplication	$g_1(t)g_2(t)$	$G_1(f) * G_2(f)$
Parseval Theorem	$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)\mathrm{d}t$	$\int_{-\infty}^{\infty} G_1(f)G_2^*(f)\mathrm{d}f$
Time Domain Integration	$\int_{-\infty}^{t} g(x)\mathrm{d}x$	$\frac{G(f)}{j2\pi f} + \frac{G(0)}{2}\delta(f)$

Formula Sheets (continued)

Table of $Q(\ )$ and $\mathrm{erf}(\ )$ functions

The approximation $Q(x) \approx \frac{1}{x\sqrt{2\pi}}(1 - \frac{0.7}{x^2})e^{-x^2/2}$ may be used when $x > 2$.

x	$\mathrm{erf}(x)$	$Q(x)$	x	$\mathrm{erf}(x)$	$Q(x)$	x	$\mathrm{erf}(x)$	$Q(x)$
0.00	0	0.5	1.70	0.9838	0.04457	3.40	1	0.0003369
0.10	0.1125	0.4602	1.80	0.9891	0.03593	3.50	1	0.0002326
0.20	0.2227	0.4207	1.90	0.9928	0.02872	3.60	1	0.0001591
0.30	0.3286	0.3821	2.00	0.9953	0.02275	3.70	1	0.0001078
0.40	0.4284	0.3446	2.10	0.997	0.01786	3.80	1	7.235×10^{-5}
0.50	0.5205	0.3085	2.20	0.9981	0.0139	3.90	1	4.810×10^{-5}
0.60	0.6039	0.2743	2.30	0.9989	0.01072	4.00	1	3.167×10^{-5}
0.70	0.6778	0.242	2.40	0.9993	0.008198	4.10	1	2.066×10^{-5}
0.80	0.7421	0.2119	2.50	0.9996	0.00621	4.20	1	1.335×10^{-5}
0.90	0.7969	0.1841	2.60	0.9998	0.004661	4.30	1	8.540×10^{-6}
1.00	0.8427	0.1587	2.70	0.9999	0.003467	4.40	1	5.413×10^{-6}
1.10	0.8802	0.1357	2.80	0.9999	0.002555	4.50	1	3.398×10^{-6}
1.20	0.9103	0.1151	2.90	1	0.001866	4.60	1	2.112×10^{-6}
1.30	0.934	0.0968	3.00	1	0.00135	4.70	1	1.301×10^{-6}
1.40	0.9523	0.08076	3.10	1	0.0009676	4.80	1	7.933×10^{-7}
1.50	0.9661	0.06681	3.20	1	0.0006871	4.90	1	4.792×10^{-7}
1.60	0.9763	0.0548	3.30	1	0.0004834	5.00	1	2.867×10^{-7}