ROYAL MILITARY COLLEGE OF CANADA

FALL TERM EXAMINATIONS 2014 – 2015

PG FINAL EXAMINATION

EE501: INTRODUCTION TO THE THEORY OF STATISTICAL COMMUNICATION

Sunday, 7 December 2014, 1300 - 1600 h.

EXAMINER: G Drolet, Associate Professor

TIME: 3 hours

NOTES:

1. Hand held calculator is allowed.
2. Only the textbook (Wozencraft & Jacobs) and course notes are allowed; problem solutions are not allowed.
3. Answer only one of questions 1 or 2. Answer all of questions 3, 4 and 5.
4. Formula sheets are attached.
5. The questions have the following value:

Question #1, #2:	5 points (chapter 3)
Question #3:	15 points (chapter 4)
Question #4:	10 points (chapter 4)
Question #5:	20 points (chapters 2, 4)

6. Justify all your answers.

Answer only one of questions 1 or 2.

1. Calculate the autocorrelation function $\mathscr{R}_x(t,s) = E[x(t)x(s)]$ of the random process:

$$x(t) = r\sin(2\pi f t + \theta) + y(t)$$

where

- $f \in \mathbb{R}$, $f > 0$ is not random,
- r is a Rayleigh distributed random variable ($b > 0$),

$$p_r(\alpha) = \begin{cases} \frac{2\alpha}{b}\mathrm{e}^{-\alpha^2/b} & ; \quad \alpha \geq 0 \\ 0 & ; \quad \alpha < 0 \end{cases}$$

- θ is a uniformly distributed random variable between 0 and 2π,

$$p_\theta(\alpha) = \begin{cases} \frac{1}{2\pi} & ; \quad 0 \leq \alpha < 2\pi \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

- $y(t)$ is a wide sense stationary random process with mean and autocorelation functions respectively denoted as m_y and $\mathscr{R}_y(\tau)$,

and $y(t)$, r, θ are statistically independent. Express your answer as a function of t, s, b, $\mathscr{R}_y(\,)$, m_y.

Hint: From section §2.10.2 we have $\overline{r} = \frac{1}{2}\sqrt{b\pi}$, $\overline{r^2} = b$, $\overline{\theta} = \pi$, $\overline{\theta^2} = 4\pi^2/3$ (you need not show this). The following trigonometric identity may be useful:

$$2\sin(u)\sin(v) = \cos(u-v) - \cos(u+v)$$

2. Let $z(t)$ be a wide-sense stationary random process with mean m_z, autocorrelation function $\mathscr{R}_z(\tau)$ and power spectral density $S_z(f)$:

$$\begin{aligned} m_z &= 0 \\ \mathscr{R}_z(\tau) &= \frac{AW}{2}\mathrm{sinc}^2(\frac{W\tau}{2}) \\ S_z(f) &= \begin{cases} A + \frac{2Af}{W} & ; \quad -\frac{W}{2} \leq f \leq 0 \\ A - \frac{2Af}{W} & ; \quad 0 \leq f \leq \frac{W}{2} \\ 0 & ; \quad \text{elsewhere} \end{cases} \end{aligned}$$

The power spectral density of $z(t)$ is sketched in figure 1. As shown in figure 2, $z(t)$ is fed through a linear invariant filter (ideal low-pass filter), the frequency response of which is given by:

$$H(f) = \begin{cases} 1 & ; \quad |f| \leq \frac{W}{4} \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

The filter's output is labelled $y(t)$.

(a) Express or sketch on a labelled graph the mean function m_y and the power spectral density $S_y(f)$ of the wide-sense stationary random process $y(t)$ as functions of A and W.

Hint: Do not use the expression for $\mathscr{R}_z(\tau)$ and do not use any table of Fourier transform pairs.

(b) Express the total average power of $y(t)$ as a function of A and W.

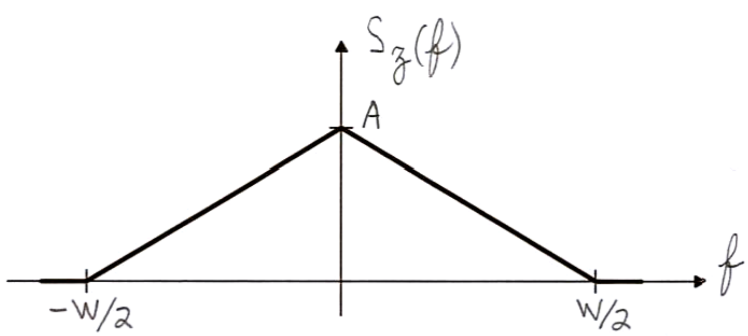


Figure 1:

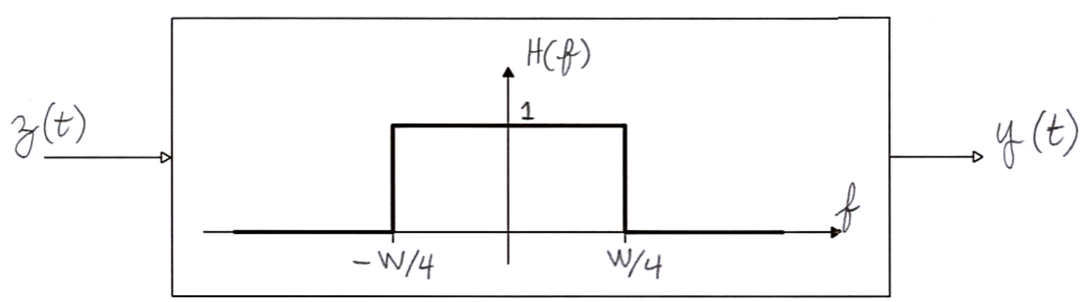


Figure 2:

FALL TERM EXAMINATIONS 2014 – 2015

EE501: INTRODUCTION TO THE THEORY OF STATISTICAL COMMUNICATION
Question 3

3. Consider a vector communication system over an additive Gaussian noise vector channel:

$$\boldsymbol{r} = \boldsymbol{s} + \boldsymbol{n},$$

where $\boldsymbol{s} = (s_1,\, s_2)$ denotes the (random) transmitted vector, $\boldsymbol{r} = (r_1,\, r_2)$ denotes the (random) received vector and as usual $\boldsymbol{n} = (n_1,\, n_2)$ is a zero-mean Gaussian vector statistically independent of $\boldsymbol{s}$, the covariance matrix of which is

$$\Lambda_{\boldsymbol{n}} = \begin{bmatrix} 0.16 & 0 \\ 0 & 0.16 \end{bmatrix}$$

The vector transmitter output is one of the four equally likely sample vectors

$$\begin{aligned} \boldsymbol{s}_0 &= (-2,\, -2) \\ \boldsymbol{s}_1 &= (2,\, -2) \\ \boldsymbol{s}_2 &= (-1,\, 1) \\ \boldsymbol{s}_3 &= (1,\, 1) \end{aligned}$$

sketched in figure 3. The decision regions of the receiver used (not necessarily optimal) correspond to the four quadrants of the plane $\mathbb{R}^2$, specifically:

$$\begin{aligned} I_0 &= \{(\rho_1,\, \rho_2) \in \mathbb{R}^2 \,:\, \rho_1 < 0 \text{ and } \rho_2 < 0\} \\ I_1 &= \{(\rho_1,\, \rho_2) \in \mathbb{R}^2 \,:\, \rho_1 > 0 \text{ and } \rho_2 < 0\} \\ I_2 &= \{(\rho_1,\, \rho_2) \in \mathbb{R}^2 \,:\, \rho_1 < 0 \text{ and } \rho_2 > 0\} \\ I_3 &= \{(\rho_1,\, \rho_2) \in \mathbb{R}^2 \,:\, \rho_1 > 0 \text{ and } \rho_2 > 0\} \end{aligned}$$

(a) Is this receiver a maximum *a posteriori* (MAP) receiver? Explain (no long calculations required; just use principles from the notes and/or textbook).

(b) Calculate $P(\mathscr{E}) \triangleq 1 - P(\mathscr{C})$ for the receiver and vector communication system described above.
Hint: From the notes we have:

$$P(\mathscr{C} \,|\, m_i) = \iint_{I_i - \boldsymbol{s}_i} p_{n_1}(\beta_1) p_{n_2}(\beta_2) \mathrm{d}\beta_1 \mathrm{d}\beta_2$$

given that the noise components $n_1,\, n_2$ are statistically independent ($\Lambda_{\boldsymbol{n}}$ is a diagonal matrix). The four regions $I_i - \boldsymbol{s}_i$, $i = 0,\, 1,\, 2,\, 3$ correspond to the hatched regions in figure 4.

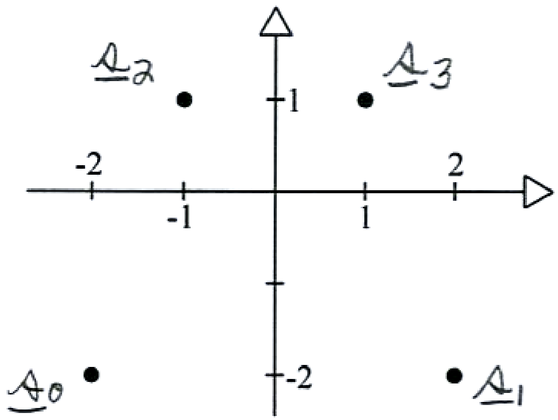
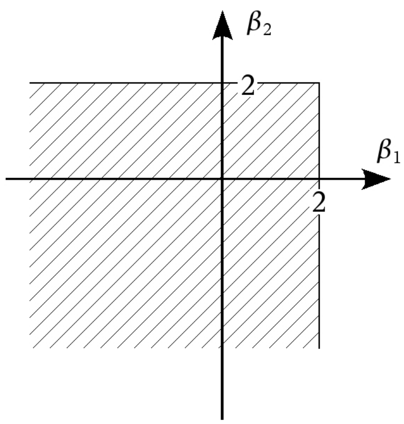
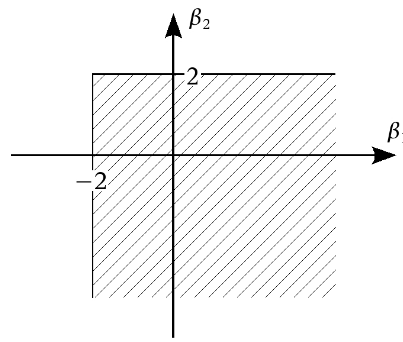


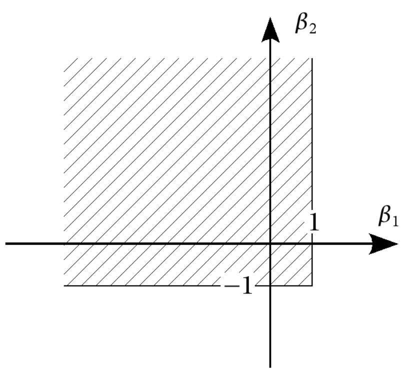
Figure 3:



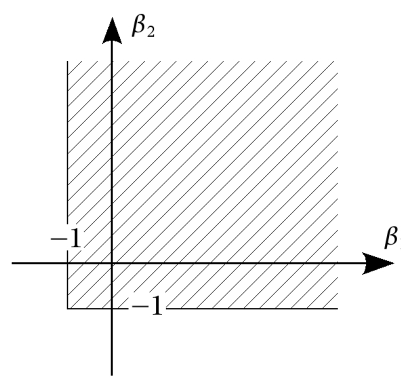
(a) $I_0 - \boldsymbol{s}_0$



(b) $I_1 - \boldsymbol{s}_1$



(c) $I_2 - \boldsymbol{s}_2$



(d) $I_3 - \boldsymbol{s}_3$

Figure 4:

4. The three equally likely signal vectors

$$\begin{aligned} \boldsymbol{s}_0 &= (-0.25\,,\,0.6) \\ \boldsymbol{s}_1 &= (1.0\,,\,-0.2) \\ \boldsymbol{s}_2 &= (-0.75\,,\,-0.4) \end{aligned}$$

are used to communicate over an additive Gaussian noise vector channel. The mean vector and covariance matrix of the noise vector $\boldsymbol{n}$ are:

$$\boldsymbol{m}_{\boldsymbol{n}} = (0\,,\,0)$$

$$\Lambda_{\boldsymbol{n}} = \begin{pmatrix} 0.16 & 0 \\ 0 & 0.16 \end{pmatrix}$$

(a) Calculate the signal-to-noise ratio.

(b) Use the *union bound* to obtain an upper bound of the probability of error. **Careful:** The signal constellation is not symmetrical. The expression $P(\mathscr{E}) = Q\left(\frac{d/2}{\sqrt{\mathscr{N}_0/2}}\right)$ may be useful.

5. This problem is based on problem 2.39 in W&J.
Warning: The expression in part 5b below should not be used to draw any conclusions about the behaviour of the suboptimum receiver by varying $\frac{E}{\sigma^2}$ or N. A bound such as that derived in problem 2.39 of W&J is much more useful than the approximation obtained.

One of two equally likely messages is transmitted over a noisy channel by means of the following strategy. If m_0 is the message, the transmitter sends a sequence of N voltage pulses over the channel, each with amplitude $\sqrt{E}$. If m_1 is the message, N voltage pulses with amplitude $-\sqrt{E}$ are sent. The effect of the channel is to add a (different) statistically independent Gaussian random variable to each amplitude. Thus the channel output is a sequence of N amplitudes

$$r_i = s + n_i; \qquad i = 1, 2, \ldots, N\,,$$

where $s = +\sqrt{E}$ if the message is m_0 and $s = -\sqrt{E}$ otherwise. Assume for all i that

$$\overline{n_i} = 0; \qquad \overline{n_i^2} = \sigma^2\,.$$

Question 5

(a) The receiver calculates

$$y = \sum_{i=1}^{N} r_i$$

and sets $\widehat{m} = m_0$ if and only if $y > 0$. [Such a receiver is optimum.] Determine the resulting $P[\mathscr{E}]$ and show that

$$P[\mathscr{E}] = Q\left(\sqrt{\frac{N\,E}{\sigma^2}}\right).$$

Suggestion: You may start from appropriate formulae in Chapter 4 of W&J (also in the notes).

(b) A suboptimum receiver makes an independent binary decision about m on the basis of each r_i in turn. Let p denote the minimum attainable probability that any such decision is wrong. Obtain an expression for p. The receiver then forms the sum

$$x = \frac{1}{N}\sum_{i=1}^{N} x_i\,,$$

where $x_i \triangleq -1$ if the ith decision favors m_1 and $x_i \triangleq +1$ if the ith decision favors m_0. Thus $P[x_i = 1|m_1] = p$ for all i independently. The receiver sets $\widehat{m} = m_0$ if $x > 0$ and $\widehat{m} = m_1$ if $x \le 0$. Using the approximation of equation (2.22) on page 72 in the notes (Central Limit Theorem), show that

$$P[\mathscr{E}] \approx Q\left(\frac{\sqrt{N}\,(1-2p)}{\sqrt{4p(1-p)}}\right)$$

Remark: By symmetry $P(\mathscr{E}|m_0) = P(\mathscr{E}|m_1)$; you need not show this.

(c) Using $\frac{E}{\sigma^2} = 0.09$ for the suboptimum receiver of part 5b, calculate the value of p and obtain an expression for $P(\mathscr{E})$ as a function of N. Then use the formula for $P(\mathscr{E})$ in part 5a to find the required value of $\frac{E}{\sigma^2}$ that would yield the same probability of error.

END

Formula Sheets

$\cos\theta = \sin(\theta + 90^\circ)$

$2\cos u \cos v = \cos(u-v) + \cos(u+v)$

$2\sin u \sin v = \cos(u-v) - \cos(u+v)$

$2\sin u \cos v = \sin(u-v) + \sin(u+v)$

$2\sin\theta\cos\theta = \sin(2\theta)$

$2\cos^2\theta = 1 + \cos(2\theta)$

$2\sin^2\theta = 1 - \cos(2\theta)$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

$P(AB) = P(A|B)P(B) = P(B|A)P(A)$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$p_x(\alpha) = \frac{\mathrm{d}F_x(\alpha)}{\mathrm{d}\alpha}$

$F_x(\alpha) = P(\{x \le \alpha\}) = \int_{-\infty}^{\alpha} p_x(u)\,\mathrm{d}u$

$P(a < x \le b) = \int_a^b p_x(\alpha)\mathrm{d}\alpha$

$p_x(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\alpha-\mu)^2/(2\sigma^2)}$

$$\begin{aligned} Q(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\beta^2/2}\mathrm{d}\beta \\ &= \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{\alpha}{\sqrt{2}}\right)\right] \\ &= 1 - Q(-\alpha) \end{aligned}$$

$$\begin{aligned} \operatorname{erf}(\alpha) &= \frac{2}{\sqrt{\pi}} \int_0^{\alpha} e^{-\beta^2}\mathrm{d}\beta \\ &= 1 - 2Q(\sqrt{2}\alpha) \end{aligned}$$

$P(x > a) = Q\left(\frac{a-\mu}{\sigma}\right)$

$P(a < x \le b) = Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right)$

$P(x \le a) = Q(\frac{\mu-a}{\sigma})$

$\frac{\mathrm{d}Q(\alpha)}{\mathrm{d}\alpha} = \frac{-1}{\sqrt{2\pi}} e^{-\alpha^2/2}$

$p_x(\alpha) = \int_{-\infty}^{\infty} p_{xy}(\alpha,\beta)\mathrm{d}\beta$

$y = bx + a \Rightarrow p_y(\alpha) = \frac{1}{|b|} p_x(\frac{\alpha-a}{b})$

$$p_{g(x)}(\beta) = p_y(\beta) = \begin{cases} \sum\limits_{\alpha \in S(\beta)} \frac{p_x(\alpha)}{|g'(\alpha)|} & ; \text{ if } S(\beta) \ne \emptyset \text{ and} \\ & \quad g'(\alpha) \ne 0, \forall \alpha \in S(\beta) \triangleq \{\alpha \in \mathbb{R} : \beta = g(\alpha)\} \\ 0 & ; \text{ if } S(y) = \emptyset \end{cases}$$

$p_{\boldsymbol{f}(\boldsymbol{x})}(\boldsymbol{\beta}) = p_{\boldsymbol{y}}(\boldsymbol{\beta}) = p_{\boldsymbol{x}}(\boldsymbol{g}(\boldsymbol{\beta}))|J_{\boldsymbol{g}}(\boldsymbol{\beta})|$

$p_x(\alpha | y = v) = p_{x|y}(\alpha, v) = \frac{p_{xy}(\alpha,v)}{p_y(v)}$

x, y are independent $\Leftrightarrow p_{xy}(\alpha,\beta) = p_x(\alpha)p_y(\beta)$

Formula Sheets (continued)

Fourier Transform Properties

Operation	$g(t)$	$G(f)$
Addition	$g_1(t) + g_2(t)$	$G_1(f) + G_2(f)$
Multiplication by a constant	$ag(t)$	$aG(f)$
Symmetry	$G(t)$	$g(-f)$
Scaling	$g(at)$	$\frac{1}{\lvert a \rvert} G(\frac{f}{a})$
Time shifting	$g(t - t_0)$	$e^{-j2\pi f t_0} G(f)$
Frequency Shifting	$e^{j2\pi f_0 t} g(t)$	$G(f - f_0)$
Modulation	$2g(t)\cos(2\pi f_c t)$	$G(f - f_c) + G(f + f_c)$
Time Differentiation	$\frac{\mathrm{d}^k g(t)}{\mathrm{d}t^k}$	$(j2\pi f)^k G(f)$
Frequency Differentiation	$(-j2\pi t)^n g(t)$	$\frac{\mathrm{d}^n G(f)}{\mathrm{d}f^n}$
Complex Conjugate	$g^*(t)$	$G^*(-f)$
Time Domain Convolution	$g_1(t) * g_2(t)$	$G_1(f) G_2(f)$
Time Domain Multiplication	$g_1(t) g_2(t)$	$G_1(f) * G_2(f)$
Parseval Theorem	$\int_{-\infty}^{\infty} g_1(t) g_2^*(t) \mathrm{d}t$	$\int_{-\infty}^{\infty} G_1(f) G_2^*(f) \mathrm{d}f$
Time Domain Integration	$\int_{-\infty}^{t} g(x) \mathrm{d}x$	$\frac{G(f)}{j2\pi f} + \frac{G(0)}{2}\delta(f)$

Formula Sheets (continued)

Table of $Q(\,)$ and $\mathrm{erf}(\,)$ functions

The approximation $Q(x) \approx \frac{1}{x\sqrt{2\pi}}(1 - \frac{0.7}{x^2})e^{-x^2/2}$ may be used when $x > 2$.

x	$\mathrm{erf}(x)$	$Q(x)$	x	$\mathrm{erf}(x)$	$Q(x)$	x	$\mathrm{erf}(x)$	$Q(x)$
0.00	0	0.5	1.70	0.9838	0.04457	3.40	1	0.0003369
0.10	0.1125	0.4602	1.80	0.9891	0.03593	3.50	1	0.0002326
0.20	0.2227	0.4207	1.90	0.9928	0.02872	3.60	1	0.0001591
0.30	0.3286	0.3821	2.00	0.9953	0.02275	3.70	1	0.0001078
0.40	0.4284	0.3446	2.10	0.997	0.01786	3.80	1	7.235×10^{-5}
0.50	0.5205	0.3085	2.20	0.9981	0.0139	3.90	1	4.810×10^{-5}
0.60	0.6039	0.2743	2.30	0.9989	0.01072	4.00	1	3.167×10^{-5}
0.70	0.6778	0.242	2.40	0.9993	0.008198	4.10	1	2.066×10^{-5}
0.80	0.7421	0.2119	2.50	0.9996	0.00621	4.20	1	1.335×10^{-5}
0.90	0.7969	0.1841	2.60	0.9998	0.004661	4.30	1	8.540×10^{-6}
1.00	0.8427	0.1587	2.70	0.9999	0.003467	4.40	1	5.413×10^{-6}
1.10	0.8802	0.1357	2.80	0.9999	0.002555	4.50	1	3.398×10^{-6}
1.20	0.9103	0.1151	2.90	1	0.001866	4.60	1	2.112×10^{-6}
1.30	0.934	0.0968	3.00	1	0.00135	4.70	1	1.301×10^{-6}
1.40	0.9523	0.08076	3.10	1	0.0009676	4.80	1	7.933×10^{-7}
1.50	0.9661	0.06681	3.20	1	0.0006871	4.90	1	4.792×10^{-7}
1.60	0.9763	0.0548	3.30	1	0.0004834	5.00	1	2.867×10^{-7}