ROYAL MILITARY COLLEGE OF CANADA

FALL TERM EXAMINATIONS 2011 – 2012

PG FINAL EXAMINATION

EE501: INTRODUCTION TO THE THEORY OF STATISTICAL COMMUNICATION

Friday, 9 December 2011, 0900 - 1200 h. (or whenever)

EXAMINER: G Drolet, Associate Professor

TIME: 3 hours

NOTES:

1. Hand held calculator is allowed.
2. Only the textbook (Wozencraft & Jacobs), course notes and personal notes are allowed; problem solutions are not allowed.
3. Answer all five questions.
4. The questions have the following value:
   Question #1: 5 points (chapter 2)
   Question #2: 7 points (chapter 3)
   Question #3: 8 points (chapter 3)
   Question #4: 15 points (chapter 4)
   Question #5: 15 points (chapter 4)
5. A table of the $Q(\ )$ and erf( ) functions is attached.
6. Justify all your answers.

1. [1] A communication system is used to transmit one of two equally likely messages, m_0 and m_1. The channel output is a continuous random variable r, the conditional density functions of which are shown in Fig. 1. Determine the optimum receiver decision rule and compute the resulting probability of error.

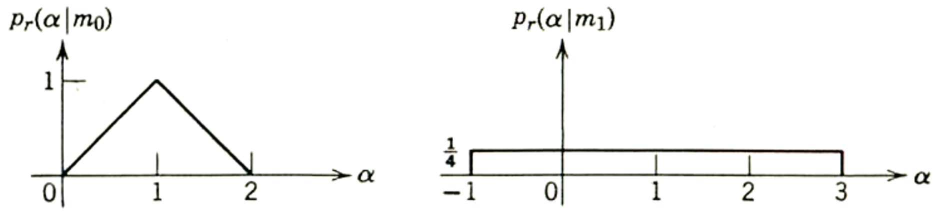

Figure 1:

2. Let $n_w(t)$ be a white Gaussian noise with power spectral density and autocorrelation function respectively given by:

$$\begin{aligned} S_w(f) &= \frac{\mathscr{N}_0}{2}, \ \forall f \\ \mathscr{R}_w(\tau) &= \frac{\mathscr{N}_0}{2}\delta(\tau) \end{aligned}$$

Two random variables y_1, y_2 are defined as follows:

$$y_1 = \int_{-\infty}^{\infty} h_1(t) n_w(t) \mathrm{d}t$$

$$y_2 = \int_{-\infty}^{\infty} h_2(t) n_w(t) \mathrm{d}t$$

where $h_1(t)$, $h_2(t)$ are two signals satisfying:

$$\begin{aligned} \int_{-\infty}^{\infty} h_1(t)^2 \mathrm{d}t &< \infty \\ \int_{-\infty}^{\infty} h_2(t)^2 \mathrm{d}t &< \infty \\ \int_{-\infty}^{\infty} h_1(t) h_2(t) \mathrm{d}t &= 0 \end{aligned}$$

Show that y_1, y_2 are statistically independent.
Hint: You may interchange expectations and integrals.

[1] Problem 2.23 in the textbook by Wozencraft & Jacobs.

3. A stationary Gaussian process $x(t)$ with mean function $m_x = 5$ and power spectral density

$$S_x(f) = \begin{cases} 2 & ; \quad -4 < f < -3 \\ 2 + 25\,\delta(f) & ; \quad -1 < f < 1 \\ 2 & ; \quad 3 < f < 4 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

is fed through two linear invariant filters as shown in figure 2. The filters frequency responses are:

$$H_z(f) = \begin{cases} 1 & ; \quad |f| > 1.5 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

$$H_y(f) = \begin{cases} 1 - |f|/2 & ; \quad |f| < 2 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

(a) Calculate m_y, m_z.
Hint: The impulse responses $h_y(t)$, $h_z(t)$ are not required; m_y and m_z can be obtained directly with the frequency responses $H_y(f)$, $H_z(f)$.

(b) Sketch $S_y(f)$, $S_z(f)$, $S_{yz}(f)$.

(c) Are the processes $y(t)$, $z(t)$ statistically independent?

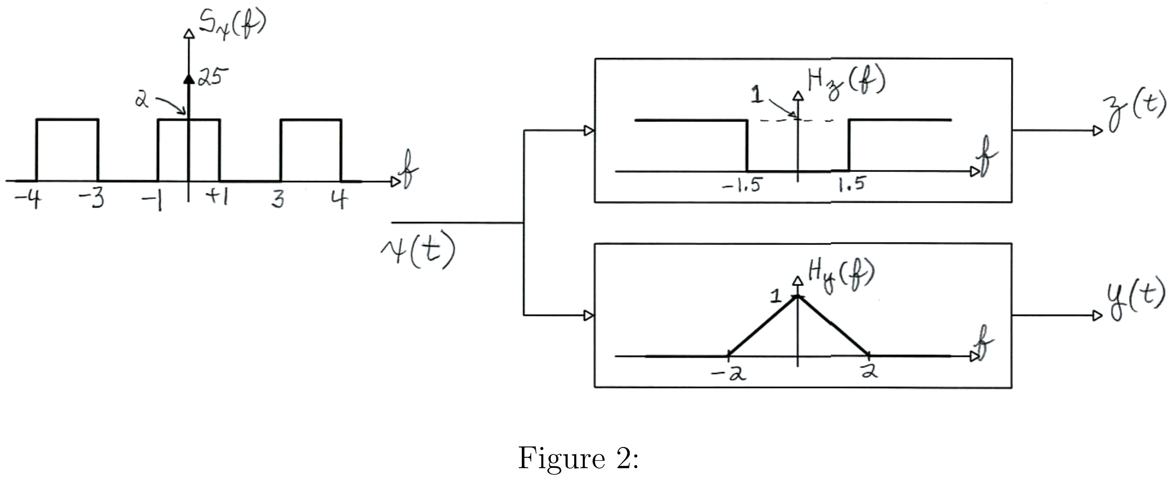

Figure 2:

FALL TERM EXAMINATIONS 2011 – 2012

EE501: INTRODUCTION TO THE THEORY OF STATISTICAL COMMUNICATION

Question 4

4. Consider a 2-dimensional additive Gaussian noise vector channel as shown in figure 4.4, page 216 in the textbook by Wozebcraft & Jacobs. The noise vector $\boldsymbol{n}$ is a 2-dimensional 0-mean Gaussian random vector which is statistically independent of $\boldsymbol{s} = (s_1,\, s_2)$. Its joint probability density function is

$$p_{\boldsymbol{n}}(\boldsymbol{\alpha}) = \frac{1}{2\pi\sqrt{|\Lambda_{\boldsymbol{n}}|}} \exp\left(\frac{-1}{2}\boldsymbol{\alpha} \times \Lambda_{\boldsymbol{n}}^{-1} \times \boldsymbol{\alpha}^{\mathrm{T}}\right)$$

where

$$\Lambda_{\boldsymbol{n}} = \begin{bmatrix} \frac{c\mathcal{N}_0}{ac-b^2} & \frac{-b\mathcal{N}_0}{ac-b^2} \\ \frac{-b\mathcal{N}_0}{ac-b^2} & \frac{a\mathcal{N}_0}{ac-b^2} \end{bmatrix}$$

$$\Lambda_{\boldsymbol{n}}^{-1} = \begin{bmatrix} a/\mathcal{N}_0 & b/\mathcal{N}_0 \\ b/\mathcal{N}_0 & c/\mathcal{N}_0 \end{bmatrix}$$

$a > 0$, $b \neq 0$, $c > 0$, $ac \neq b^2$ and the two components of $\boldsymbol{n}$ are clearly *not statistically independent*. The source sends one of two equally likely messages m_0, m_1 and the vector transmitter is defined by the mapping:

$$m = m_0 \quad \Leftrightarrow \quad \boldsymbol{s} = \boldsymbol{s}_0 = \left(\sqrt{E_s/2}, \sqrt{E_s/2}\right)$$

$$m = m_1 \quad \Leftrightarrow \quad \boldsymbol{s} = \boldsymbol{s}_1 = \left(\sqrt{E_s/2}, -\sqrt{E_s/2}\right)$$

Notice that both signal vectors have the *same projection on the first dimension.* As remarked in problem 4.12(b) in Wozencraft & Jacobs, this first component cannot be ignored by an optimal receiver since n_1 and n_2 are not statistically independent.

The MAP decision regions are (you need not show this):

$$I_0 = \left\{\boldsymbol{\rho} = (\rho_1,\, \rho_2) \in \mathbb{R}^2 : b\rho_1 + c\rho_2 > b\sqrt{E_s/2}\right\}$$

$$I_1 = \left\{\boldsymbol{\rho} = (\rho_1,\, \rho_2) \in \mathbb{R}^2 : b\rho_1 + c\rho_2 < b\sqrt{E_s/2}\right\}$$

where $\boldsymbol{r} = \boldsymbol{\rho}$ is the received vector. If $b\rho_1 + c\rho_2 = b\sqrt{E_s/2}$, either message can be chosen.

Express $P(\mathscr{E})$ as a function of the signal-to-noise ratio $E_s/\mathcal{N}_0$ and the constants a, b and c.

FALL TERM EXAMINATIONS 2011 – 2012

EE501: INTRODUCTION TO THE THEORY OF STATISTICAL COMMUNICATION

Question 5

5. [2] Either of the two signal waveform sets illustrated in Fig. 3 or 4 may be used to communicate one of four equally likely messages over an additive white Gaussian noise channel.

   (a) Show that both sets use the same average energy.

   (b) Exploit the union bound to show that the set of Fig. 4 uses energy almost 3 dB more effectively than the set of Fig. 3 when a small $P[\mathscr{E}]$ is required. Specifically, for each set

      i. calculate the distance between any two signals,

      ii. use the above distances together with the expression $Q\left(\frac{d/2}{\sqrt{\mathscr{N}_0/2}}\right)$ to calculate the union bound on the probability of error.

   Conclude by comparing the two union bounds, noticing that $Q(a) \leq Q(b)$ whenever $a \geq b$. More specifically,

$$Q\left(\sqrt{\frac{E_s}{\mathscr{N}_0}}\right) \leq Q\left(\sqrt{\frac{E_s}{2\,\mathscr{N}_0}}\right)$$

   by about 3 dB as remarked in Wozencraft & Jacobs page 251, and figure 4.11 in the notes.

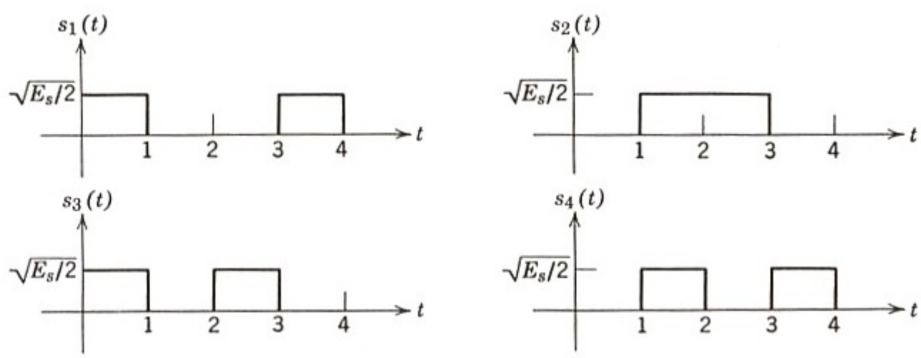

Figure 3:

[2]Problem 4.18, pages 283-284 in W & J.

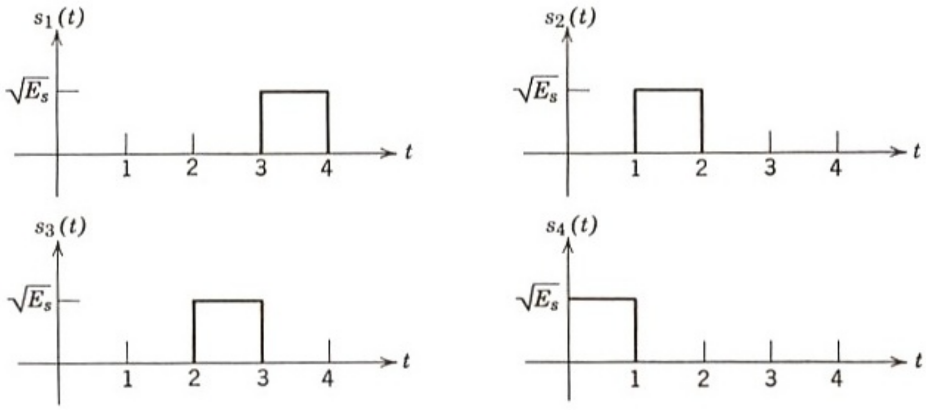
$s_1(t)$
$\sqrt{E_s}$
1
2
3
4
t
$s_2(t)$
$\sqrt{E_s}$
1
2
3
4
t
$s_3(t)$
$\sqrt{E_s}$
1
2
3
4
t
$s_4(t)$
$\sqrt{E_s}$
1
2
3
4
t

Figure 4:

END

Table of the $Q(\quad)$ and $\mathrm{erf}(\quad)$ functions

The approximation $Q(x) \approx \frac{1}{x\sqrt{2\pi}}(1 - \frac{0.7}{x^2})e^{-x^2/2}$ may be used when $x > 2$.

x	$\mathrm{erf}(x)$	$Q(x)$	x	$\mathrm{erf}(x)$	$Q(x)$	x	$\mathrm{erf}(x)$	$Q(x)$
0.00	0	0.5	1.70	0.9838	0.04457	3.40	1	0.0003369
0.10	0.1125	0.4602	1.80	0.9891	0.03593	3.50	1	0.0002326
0.20	0.2227	0.4207	1.90	0.9928	0.02872	3.60	1	0.0001591
0.30	0.3286	0.3821	2.00	0.9953	0.02275	3.70	1	0.0001078
0.40	0.4284	0.3446	2.10	0.997	0.01786	3.80	1	7.235×10^{-5}
0.50	0.5205	0.3085	2.20	0.9981	0.0139	3.90	1	4.810×10^{-5}
0.60	0.6039	0.2743	2.30	0.9989	0.01072	4.00	1	3.167×10^{-5}
0.70	0.6778	0.242	2.40	0.9993	0.008198	4.10	1	2.066×10^{-5}
0.80	0.7421	0.2119	2.50	0.9996	0.00621	4.20	1	1.335×10^{-5}
0.90	0.7969	0.1841	2.60	0.9998	0.004661	4.30	1	8.540×10^{-6}
1.00	0.8427	0.1587	2.70	0.9999	0.003467	4.40	1	5.413×10^{-6}
1.10	0.8802	0.1357	2.80	0.9999	0.002555	4.50	1	3.398×10^{-6}
1.20	0.9103	0.1151	2.90	1	0.001866	4.60	1	2.112×10^{-6}
1.30	0.934	0.0968	3.00	1	0.00135	4.70	1	1.301×10^{-6}
1.40	0.9523	0.08076	3.10	1	0.0009676	4.80	1	7.933×10^{-7}
1.50	0.9661	0.06681	3.20	1	0.0006871	4.90	1	4.792×10^{-7}
1.60	0.9763	0.0548	3.30	1	0.0004834	5.00	1	2.867×10^{-7}